Exercise sheet 2

Algebraic Topology

October 17, 2023

Exercise 1. A singular 1-simplex $\sigma\colon [0,1] \to X$ is called a loop if $\sigma(0) = \sigma(1)$.

1. Prove that a loop is an 1-cycle.
2. Two loops are called freely homotopic if there is a continuous map $F\colon [0,1] \times [0,1] \to X$ such that $F(0,t) = \sigma_0(t)$, $F(1,t) = \sigma_1(t)$, and each $F(s,\ldots)$ is a loop. Show that free homotopy defines an equivalence relation on the sets of loops in X.
3. Show that two freely homotopic loops are homologous, i.e. they define the same element in $H_1(X)$. *Hint: Draw a picture and try to triangularize it.*
4. A 1-chain $\sigma_0 + \ldots + \sigma_{r-1}$ with $\sigma_i(1) = \sigma_{i-1}(1)$ for all $i \in \mathbb{Z}_r$ is called an elementary 1-cycle. Prove that an elementary 1-cycle is a 1-cycle and it is homologous to a loop.
5. Prove that the class of elementary 1-cycles generate $H_1(X)$.

Exercise 2. Let A be a subspace of the topological space X.

1. Assume that there exists a map $r\colon X \to A$ such that $r|_A$ is the identity map (in that case we call r a *retraction* map and A a *retract* of X). For any $k \in \mathbb{Z}$, let r_k be the induced map of r between $H_k(X)$ to $H_k(A)$. Show that

$$H_k(X) \simeq H_k(A) \oplus \ker r_k.$$

2. Assume that there exists a map $R\colon X \times [0,1] \to X$ such that

   - $R(a,t) = a$ for all $a \in A$ and $t \in [0,1]$,
   - $R(x,0) = x$ for all $x \in X$, and
   - $R(x,1) \in A$ for all $x \in X$

   (in that case we call R a *deformation retraction map* and A a deformation retract of X). Show that $H_k(X)$ is isomorphic to $H_k(A)$ for all $k \in \mathbb{Z}$.

Exercise 3. Let X and Y be topological spaces and let $f_n, g_n : S_n(X) \to S_n(Y)$ be chain maps. We say f_n and g_n are *chain homotopic* if there exists a family of maps $P_n : C_n(X) \to C_{n+1}(Y)$ such that

$$\partial_{n+1} P_n - P_{n-1}\partial_n = f_n - g_n.$$

We say that the spaces X and Y are *chain homotopic* if there are chain maps $f_n : S_n(X) \to S_n(Y)$ and $g_n : S_n(Y) \to S_n(X)$ such that $f_n \circ g_n$ and $g_n \circ f_n$ are chain homotopic to the identity maps.

1. Show that if f_n is chain homotopic to g_n, then the induced maps f_*, g_* in homology are equal.

2. Show that if X is chain homotopic to Y, then $H_k(X) \simeq H_k(Y)$ for all $k \in \mathbb{Z}$.

Exercise 4. See Exercise 2.35 in the lecture notes: Show that the Bockstein homomorphism ∂ is a group homomorphism.

Exercise 5. Let X be a topological space equipped with a triangulation C_k^Δ. The simplicial homology of X with real coefficients $H_k^\Delta(X, \mathbb{R})$ is computed as the ordinary simplicial homology, but taking combination with real coefficients instead of integer coefficients. In particular the n-chains with real coefficients are a real vector space and the boundary maps are linear maps.The Euler characteristic of X is defined as

$$\chi(X) = \sum_k (-1)^k \dim H_k^\Delta(X, \mathbb{R}).$$

Show that $\chi(X) = \sum_k (-1)^k \dim C_i^\Delta(X, \mathbb{R})$. Compute the Euler characteristic of S^2 and T^2.

Cool fact 1. If you have a compact surface Σ you can actually calculate $\chi(X)$ by integration over a certain function, called the Gauss curvature. This result, called the Gauss-Bonnet theorem, is one of the magical relations between algebraic topology and differential geometry.

Cool fact 2. There are generalisations of the Gauss-Bonnet theorem that actually pop up in particle physics!