# Exercise sheet 4

Algebraic Topology

November 7, 2023

**Exercise 1.** Let $X$ be a topological space. The **suspension** $SX$ of $X$ is the space $[-1,1] \times X$, quotiented by the relationship

$$(s,x) \sim (t,y) \Leftrightarrow s = t \text{ and } s,t \in \{\pm 1\}.$$

Show that

- $H_k(X) \simeq H_{k+1}(SX)$ for all $k \geq 1$,
- $H_0(X) \simeq H_1(SX) \oplus \mathbb{Z}$, and
- $H_0(SX) \simeq \mathbb{Z}$.

**Exercise 2.** Prove Theorem 2.72 from the lecture notes. That is, show that the homology groups of the Klein bottle are

$$H_k(K) \simeq \begin{cases} \mathbb{Z} & \text{for } k = 0, \\ \mathbb{Z} \oplus \mathbb{Z}_2 & \text{for } k = 1, \\ 0 & \text{else.} \end{cases}$$

**Exercise 3.** Let $X$ and $Y$ be path-connected topological spaces, $x \in X$ and $y \in Y$. The **wedge sum** or the one-point union $X \vee Y$ is the disjoint union of $X$ and $Y$, quotiented by the relation ship that identifies $x$ to $Y$. Show that if $x$ and $Y$ have contractible neighbourhoods in $X$ and $Y$ respectively,

$$H_k(X \vee Y) = \begin{cases} H_k(X) \oplus H_k(Y) & \text{if } k \geq 1, \\ \mathbb{Z} & \text{if } k = 0. \end{cases}$$

**Exercise 4.** Solve the "Three utilities problem": Suppose there are three cottages on a plane and each needs to be connected to the water, gas, and electricity companies. Without using a third dimension or sending any of the connections through another company or cottage, is there a way to make all nine connections without any of the lines crossing each other?
*See Exercise 2.69 in the lecture notes for a hint.*