Exercise sheet 6

Algebraic Topology

November 21, 2023

Exercise 1. Let X be the quotient space of S^2 under the identifications $x \sim -x$ for x in the equator S^1. Show that the cellular homology of X equals

$$H_k(X) \simeq \begin{cases} \mathbb{Z} & \text{for } k = 0, 2 \\ \mathbb{Z}_2 & \text{for } k = 1, \\ 0 & \text{else.} \end{cases}$$

Optional: Do the same for S^3 with the antipodal points of the equatorial $S^2 \subset S^3$ identified.

Exercise 2. Let Σ_g be the compact surface of genus g (See Section 2.11.5). Without using Theorem 2.82, show that the Euler characteristic of Σ_g is

$$\chi(\Sigma_g) = 2 - 2g.$$

Exercise 3. Let X be a topological space and let $f\colon X \to X$ be a continuous map. The mapping torus T_f is the quotient of the space $X \times [0, 1]$ under the relation

$$(x, 0) \sim (f(x), 1).$$

Find a suitable set $B \subset T_f$, such that $H_{k+1}(T_f, B) \simeq H_k(X)$ and show that the long exact sequence for relative homology can be written as

$$\ldots \to H_k(X) \xrightarrow{1-f_*} H_k(X) \to H_k(T_f) \to H_{k-1}(X) \to \ldots$$

Exercise 4. Prove the Borsuk-Ulam theorem. That is, show that for any continuous map $f\colon S^n \to \mathbb{R}^n$, there exists an $x \in S^n$ such that $f(x) = f(-x)$. (Colloquially, this theorem implies that there is a pair of antipodal points on the earth, with the same temperature and pressure.)