Exercise sheet 7

Algebraic Topology

November 28, 2023

Exercise 1. Recall from Example 3.8, 3.9 and 3.31 that there is a natural way to embed $\mathbb{C}P^n$ and $\mathbb{R}P^n$ inside $\mathbb{C}P^{n+1}$ and $\mathbb{R}P^{n+1}$ respectively. Hence define

$$\mathbb{C}P^\infty = \bigcup_{n\in\mathbb{N}} \mathbb{C}P^n \text{ and } \mathbb{R}P^\infty = \bigcup_{n\in\mathbb{N}} \mathbb{R}P^n.$$

Equip $\mathbb{C}P^\infty$ and $\mathbb{R}P^\infty$ with a CW-complex and show that

$$H_k(\mathbb{C}P^\infty) = \begin{cases} \mathbb{Z} & k \equiv 0 \mod 2 \\ 0 & k \equiv 1 \mod 2, \end{cases}$$

and

$$H_k(\mathbb{R}P^\infty) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}_2 & k \equiv 1 \mod 2, \\ 0 & \text{else.} \end{cases}$$

Exercise 2. Compute the homology groups of the following 2-complexes:

1. The quotient of S^2 obtained by identifying the north and south pole to a point.
2. $S^1 \times (S^1 \vee S^1)$
3. The space obtained from a closed disk D^2 by first deleting the interiors of two distjoint subdisks in the interior of D^2 and then identifying all the three resulting boundary circles together via homeomorphisms preserving clockwise orientations of these circles.
4. The quotient space $S^1 \times S^1$ obtained by identifying points in the circle $S^1 \times \{y_0\}$ that differ by $2\pi/m$ rotation and identifying points in the circle $\{x_0\} \times S^1$ that differ by $2\pi/n$ rotation.

Exercise 3. Let X be the quotient space of S^2 under the identifications $x \sim -x$ for x in the equator S^1. Show that the cellular homology of X equals

$$H_k(X) \simeq \begin{cases} \mathbb{Z} & \text{for } k = 0, 2 \\ \mathbb{Z}_2 & \text{for } k = 1, \\ 0 & \text{else.} \end{cases}$$

Do the same for S^3 with the antipodal points of the equatorial $S^2 \subset S^3$ identified.