Exercise sheet 9

Algebraic Topology

December 12, 2023

Exercise 1. See Corollary 4.16. Show that the fundamental group of T^2 is isomorphic to $\mathbb{Z} \times \mathbb{Z}$ and that the fundamental group of $\mathbb{R}P^n$ is $\mathbb{Z}_2$ for $n \geq 2$.

Exercise 2. Let X be a topological space that is the union of two path-connected open sets A and B. Assume that there exists an $x_0 \in A \cap B$ and that $A \cap B$ is path-connected.

1. Show that there is a surjective group homomorphism $\Phi \colon \pi_1(A, x_0) * \pi_1(B, x_0) \to \pi_1(X, x_0)$, where $*$ denotes the free group product.
2. Conclude that $\pi_1(S^1 \vee S^1) = \frac{\mathbb{Z} * \mathbb{Z}}{G}$ where G is a normal subgroup of $Z * \mathbb{Z}$.
3. In the next exercise we will show $\pi_1(S^1 \vee S^1) = \pi_1(S^1) * \pi_1(S^1)$. Can you give an example where $\pi_1(X, x_0) \neq \pi_1(A, x_0) * \pi_1(B, x_0)$? What could be the issue? (For comparison, when is $H_1(X) \neq H_1(A) \oplus H_1(B)$?)

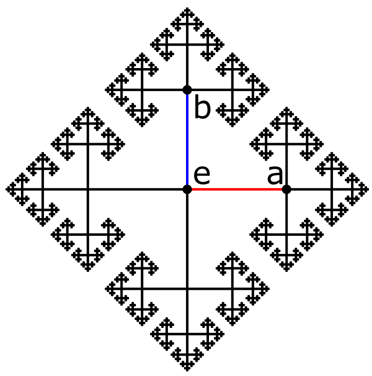

Figure 1: Universal cover of of $S^1 \vee S^1$

Exercise 3. Let X be the infinite CW-complex[1] depicted in figure 1.

1. Show that X is path-connected and simply connected.

2. On the set of vertices of X, one can define the following $\langle a, b\rangle = \mathbb{Z} * \mathbb{Z}$ action using the following rules:

   - We start with the empty word e.
   - Every time we traverse one edge to the right, we add the letter a to our word.
   - Every time we traverse one edge to the left, we add the letter a^{-1} to our word.
   - Every time we traverse one edge up, we add the letter b to our word.
   - Every time we traverse one edge down, we add the letter b^{-1} to our word.

   Show that this $\mathbb{Z} * \mathbb{Z}$ action extends to an action on X.

3. Show that $X/\mathbb{Z} * \mathbb{Z}$ is homotopic to $S^1 \vee S^1$.
   Hint. How does $\mathbb{Z} * \mathbb{Z}$ act on the CW-structure of X?

4. Show that X is the universal cover of $S^1 \vee S^1$.

5. Prove that $\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}$.

Exercise 4. We can regard $\pi_1(X, x_0)$ as the set of basepoint-preserving homotopy classes of maps $(S^1, x_0) \to (X, x_0)$. Let $[S^1, X]$ be the set of homotopy classes of maps $S^1 \to X$, with no conditions on the basepoints. Thus, there is a natural map $\Phi\colon \pi_1(X, x_0) \to [S^1, X]$ obtained by ignoring basepoints. Show that Φ is onto if X is path-connected, and that $\Phi([f]) = \Phi([g])$ iff $[f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$.

[1] We do not require that X has the induced subspace-topology of $\mathbb{R}^2$.