Differential Geometry I

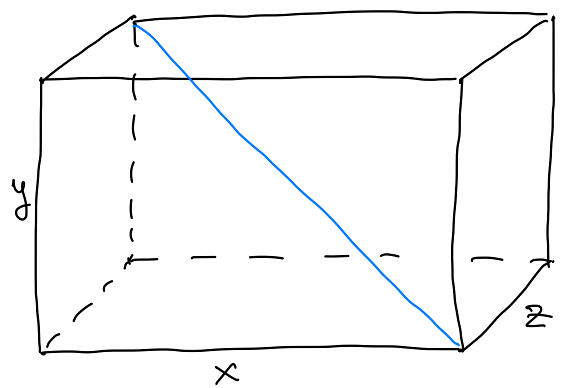
Let $U \subset \mathbb{R}^n$ be an open subset and $f \in C^1(U)$. It is known from analysis that $x_0 \in U$ is a point of extremum for f if

$$\frac{\partial f}{\partial x_i}(x_0) = 0 \qquad \forall i = 1, \dots, n.$$

Notice that this is a necessary condition, which is not sufficient in general.

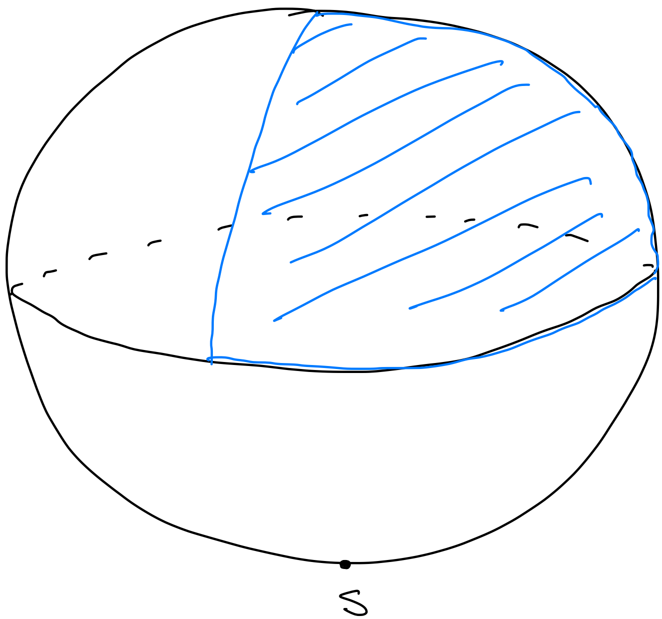
A more general type of problems does not fit into this scheme. For example, consider the following.

Problem Among all rectangular parallelepipeds, whose diagonal has a fixed length 1, find the one with maximal volume.



Thus, we want to find a point of maximum of the function $f(x,y,z) = xyz$ on the set 2

$V = \{(x,y,z) \in \mathbb{R}^3 \mid x>0, y>0, z>0 \text{ and } x^2+y^2+z^2 = 1\} \subset S^2$



However, V is not an open subset of $\mathbb{R}^3$ so that the receipy known from the analysis course is not applicable.

This problem is relatively easy to solve, however. Indeed, since $z>0 \Rightarrow z = \sqrt{1-x^2-y^2}$ so that

$$f(x,y,\sqrt{1-x^2-y^2}) = \underbrace{xy\sqrt{1-x^2-y^2}}_{\overset{!!}{F(x,y)}}, \qquad x^2+y^2<1$$

Hence, we want to find points of maximum of the function F on the set $\{ (x,y) \mid x^2+y^2<1,\ x>0,\ y>0 \}$, which is an open subset of $\mathbb{R}$.

We compute

$$\frac{\partial F}{\partial x} = y\sqrt{1-x^2-y^2} - xy\frac{x}{\sqrt{1-x^2-y^2}} = 0$$

$$(*)$$

$$\frac{\partial F}{\partial y} = x\sqrt{1-x^2-y^2} - xy\frac{y}{\sqrt{1-x^2-y^2}} = 0$$

Since $x \neq 0$ and $y \neq 0$, we have

$$(*) \iff \begin{array}{l} 1-x^2-y^2 = x^2 \\ 1-x^2-y^2 = y^2 \end{array} \Longrightarrow x^2 = y^2 \Longrightarrow x = y$$

$$\Longrightarrow 3x^2 = 1 \Longrightarrow x = y = \frac{1}{\sqrt{3}}$$

$$\Longrightarrow z = \frac{1}{\sqrt{3}}$$

Hence, among all rectangular parallelepipeds with the given length of the diagonal the cube maximazes the volume.

<u>Exercise</u> Show that $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ is a point of maximum indeed.

Consider a more general problem of constrained maximum / minimum. Given $f, \varphi \in C^\infty(\mathbb{R}^n)$ find a point of maximum / minimum of f on the set

$$S := \{ x \in \mathbb{R}^n \mid \varphi(x) = 0 \}.$$

<u>Prop</u> 1 Assume that for $p \in S$ we have

$$\frac{\partial \varphi}{\partial x_n}(p) \neq 0. \qquad (*)$$

Then $\exists$ a neighbourhood U of p in S, an open subset $V \subset \mathbb{R}^{n-1}$, and a smooth function $\psi : V \to \mathbb{R}$ such that

$$x = (\underset{\mathbb{R}^{n-1}}{\underset{\cap}{y}}, \underset{\mathbb{R}}{\underset{\cap}{z}}) \in S \cap U \iff z = \psi(y), \ y \in V.$$

This is a celebrated implicit function theorem, whose proof was given in the analysis course.

<u>Thm</u> 1 Let $p \in S$ be a point of (local) maximum of f on S. If $(*)$ holds, then $\exists \lambda \in \mathbb{R}$ such that

$$\frac{\partial f}{\partial x_j}(p) = \lambda \frac{\partial \varphi}{\partial x_j}(p) \iff \nabla f(p) = \lambda \nabla \varphi(p)$$

holds for each $j = 1, \dots, n$.

<u>Proof</u> Let $p = (y_0, z_0)$.

p is a (loc.) maximum for $f|_S \iff$

y_0 is a loc. maximum for a function

$$F : V \to \mathbb{R}, \quad F(y) := f(y, \psi(y))$$

$$\Rightarrow \frac{\partial F}{\partial y_j}(y_0) = \frac{\partial f}{\partial y_j}(p) + \frac{\partial f}{\partial x_n}(p) \cdot \frac{\partial \psi}{\partial y_j}(y_0) = 0$$

$\forall j \leq n-1$

$$\varphi(y, \psi(y)) = 0 \Rightarrow \frac{\partial \varphi}{\partial y_j} + \frac{\partial \varphi}{\partial x_n} \frac{\partial \psi}{\partial y_j} \equiv 0$$

$$\Rightarrow \frac{\partial \psi}{\partial y_j}(y_0) = -\frac{\partial \varphi}{\partial y_j}(p) \Big/ \frac{\partial \varphi}{\partial x_n}(p)$$

substitute

$$\Rightarrow \frac{\partial f}{\partial y_j}(p) = \underbrace{\left(\frac{\partial f}{\partial x_n}(p) \Big/ \frac{\partial \varphi}{\partial x_n}(p) \right)}_{= \lambda \text{ does not depend on } j} \cdot \frac{\partial \varphi}{\partial y_j}(p)$$

For $j = n$ we have

$$\frac{\partial f}{\partial x_n}(p) = \left(\frac{\partial f}{\partial x_n}(p) \Big/ \frac{\partial \varphi}{\partial x_n}(p) \right) \cdot \frac{\partial \varphi}{\partial x_n}(p) \quad \checkmark$$

$\blacksquare$

Let us come back to the example about maximal volume of parallelepipeds with a fixed length of the diagonal. Thus, if (x,y,z) is a point of maximum of f on

$$S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1,\ x,y,z>0\},$$

then there exists $\lambda \in \mathbb{R}$ such that

$$\begin{aligned} yz &= 2\lambda x \\ xz &= 2\lambda y \\ xy &= 2\lambda z \end{aligned} \quad \Rightarrow \quad (xyz)^2 = 8\lambda^3 xyz$$

$\lambda \neq 0$, since otherwise $x=0$ or $y=0$ or $z=0$.

$$\Rightarrow \quad \underbrace{xyz}_{=2\lambda x^2} = 8\lambda^3 \quad \Rightarrow \quad x = 2\lambda$$

using the first eqn

A similar argument yields also $y = 2\lambda$ and $z = 2\lambda$

$$\Rightarrow \quad 4\lambda^2 + 4\lambda^2 + 4\lambda^2 = 1 \quad \Rightarrow \quad \lambda = \frac{1}{2\sqrt{3}}$$

$$\Rightarrow \quad x = y = z = \frac{1}{\sqrt{3}}$$

Coming back to Prop. 1 on P. 4, it is clear that it is only important that one of the partial derivatives of φ does not vanish. This leads to the following definition.

<u>Def</u> (Surface) A non-empty set $S \subset \mathbb{R}^3$ is called a (smooth) surface, if for any $p \in S$ $\exists$ an open set $V \subset \mathbb{R}^2$ and a smooth map $\psi : V \to S$ such that the following holds:

(i) $\psi(V) =: U$ is a neighbourhood of p in S.

(ii) $\psi : V \to U$ is a homeomorphism.

(iii) $D_q\psi : \mathbb{R}^2 \to \mathbb{R}^3$ is injective $\forall q \in V$.

<u>Ex</u> Assume $\varphi \in C^\infty(\mathbb{R}^3)$ satisfies

$$\frac{\partial \varphi}{\partial z}(p) \neq 0 \qquad \forall p \in S = \{\varphi(x,y,z) = 0\}.$$

Let ψ be as in Prop 1 on P. 4. Define $\Psi(x,y) := (x, y, \psi(x,y))$. If U and V are also as in Prop. 1 on P4, then $\Psi : V \to S \cap U$ is a homeomorphism, since $\pi : S \cap U \to V$, $\pi(x,y,z) = (x,y)$

is a continuous inverse. Furthermore,

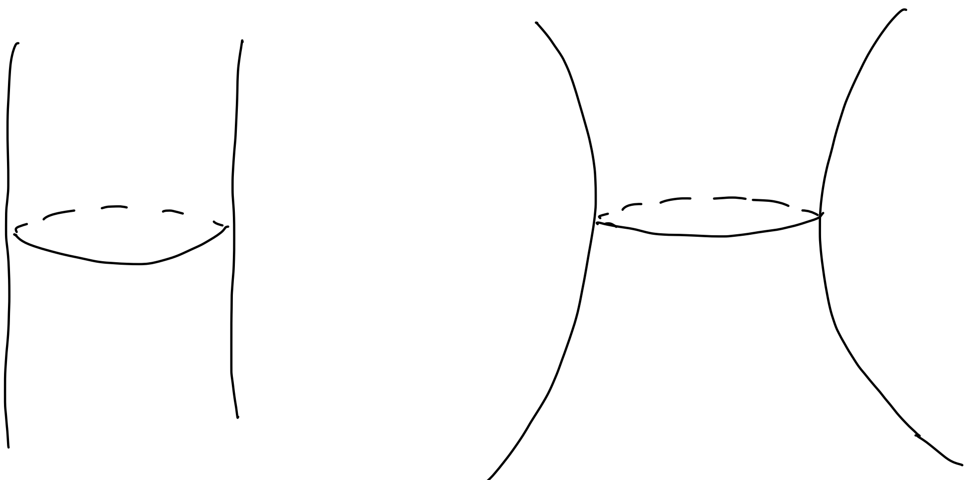
$$D\Psi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} \end{pmatrix}$$ is clearly injective at all points.

Hence, S is a surface.

In particular,

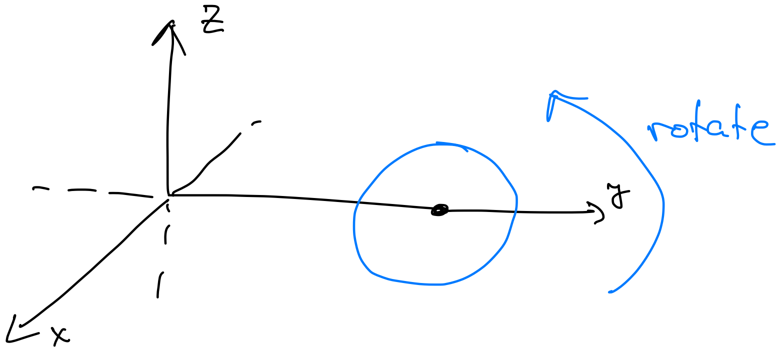
- the sphere $S^2 = \{ x^2+y^2+z^2=1 \}$
- the cylinder $C = \{ (x,y,z) \mid x^2+y^2=1 \}$
- the hyperboloid $H = \{ x^2+y^2-z^2=1 \}$

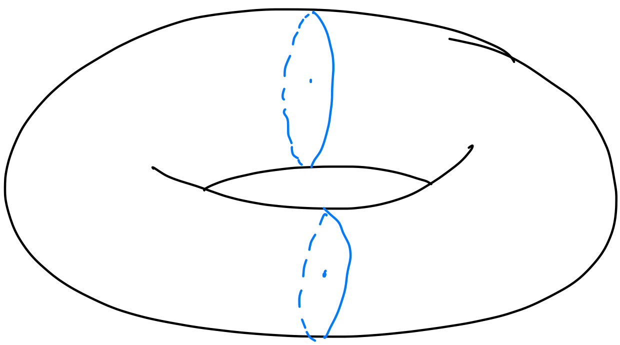
are surfaces



C H

Ex (Torus) Let C be the circle of radius r in the yz-plane centered at the point $(0, a, 0)$, where $a > r$





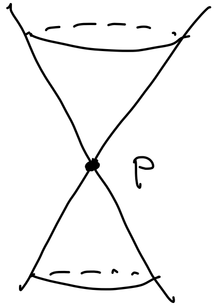
Torus

More formally,

$$T := \{ (\sqrt{x^2+y^2} - a)^2 + z^2 = r^2 \}$$

Exercise Check that T is a surface indeed.

<u>A non-example</u> A double cone



$C_0 := \{x^2 + y^2 - z^2 = 0\}$
is not a surface.

Indeed, assume C_0 is a surface. Then the tip of the cone p must have a neighbourhood U homeomorphic to an open disc in $\mathbb{R}^2$.
Let $f : U \to D$ be a homeomorphism.
Then $f : U \setminus \{p\} \to D \setminus f(p)$ is also a homeo.

$U \setminus \{p\}$: disconnected; $D \setminus f(p)$: connected

Hence, p does not have a nbhd homeomorphic to a disc (or any open subset of $\mathbb{R}^2$).

<u>Exercise</u> Show that a straight line is not a surface.

<u>Rem</u> 1) The map φ in the definition of the surface is called a <u>parametrization</u>.

2) Condition (iii) is equivalent to

$$\frac{\partial \varphi}{\partial u} \quad \& \quad \frac{\partial \varphi}{\partial v}$$

are linearly independent at each pt $(u,v) \in V$.

Prop Let S be a surface. For any $p \in S$ $\exists$ a nbhd $W \subset \mathbb{R}^3$ and $\varphi \in C^\infty(W)$ such that

$$S \cap W = \{ x \in W \mid \varphi(x) = 0 \}$$

and $\nabla \varphi(x) \neq 0 \quad \forall x \in S \cap W$.

Proof Choose a parametrization $\psi : \underset{\mathbb{R}^2}{\overset{\cap}{V}} \to \underset{S}{\overset{\cap}{U}}$.

Let $\psi(u_0, v_0) = p$ and choose a vector $n \in \mathbb{R}^3$ such that

$$\frac{\partial \psi}{\partial u}(u_0, v_0) \ , \quad \frac{\partial \psi}{\partial v}(u_0, v_0) \ , \quad n \qquad (*)$$

are linearly independent. Consider the map

$$\Psi : V \times \mathbb{R} \to \mathbb{R}^3, \quad \Psi(u, v, w) = \psi(u, v) + w \cdot n$$

By $(*)$, $\det D\Psi(u_0, v_0, 0) \neq 0$. By the inverse map theorem, $\exists$ open neighbourhood $W \subset \mathbb{R}^3$ of p and a smooth map $\Phi : W \to V \times \mathbb{R} \subset \mathbb{R}^3$ such that

$$\Psi \circ \Phi(x) = x \qquad \forall x \in W$$

If $\Phi = (\varphi_1, \varphi_2, \varphi_3)$, then

$$\Psi \circ \Phi(x) = \psi(\varphi_1(x), \varphi_2(x)) + \varphi_3(x) \cdot n = x$$

Observe that

$$x \in S \cap W \iff \exists (u,v) \in V \text{ s.t. } \varphi(u,v) = x$$

$$x = \varphi(u,v) = \Psi(u,v,0)$$
$$\| $$
$$\Psi(\varphi_1(x), \varphi_2(x), \varphi_3(x))$$

Since Ψ is injective (on an open nbhd of $(u_0, v_0, 0)$), we have

$$x \in S \cap W \iff \varphi_3(x) = 0.$$

Furthermore, $\det D\Phi(x) \neq 0 \quad \forall x \in W$

$\iff \nabla\varphi_1(x), \nabla\varphi_2(x), \nabla\varphi_3(x)$ are linearly independent $\forall x \in W$

$\implies \nabla\varphi_3(x) \neq 0 \quad \forall x \in W.$ $\blacksquare$

<u>Corollary</u> Any surface is locally the graph of a smooth function.

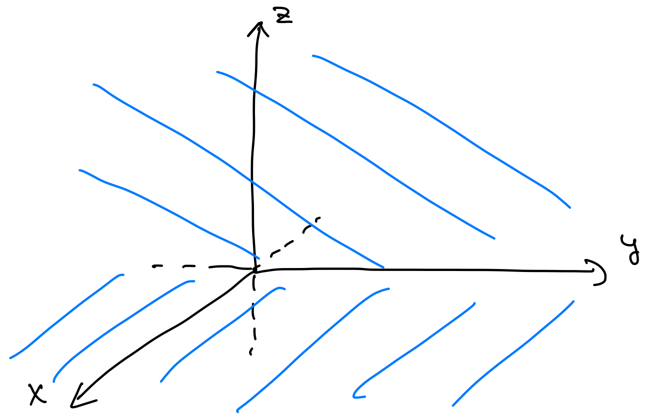
The proof follows from Prop 1 on P. 4.

<u>A non-example</u> A union of two intersecting planes is <u>not</u> a surface

Indeed, assume that

$S := \{z=0\} \cup \{x=0\}$

is a surface.



Then $\exists$ a smooth function φ defined in a nbhd W of the origin such that

$$\varphi(x,y,z)=0 \quad \text{on} \quad S$$

$$\Rightarrow \quad \nabla\varphi(0)=0.$$

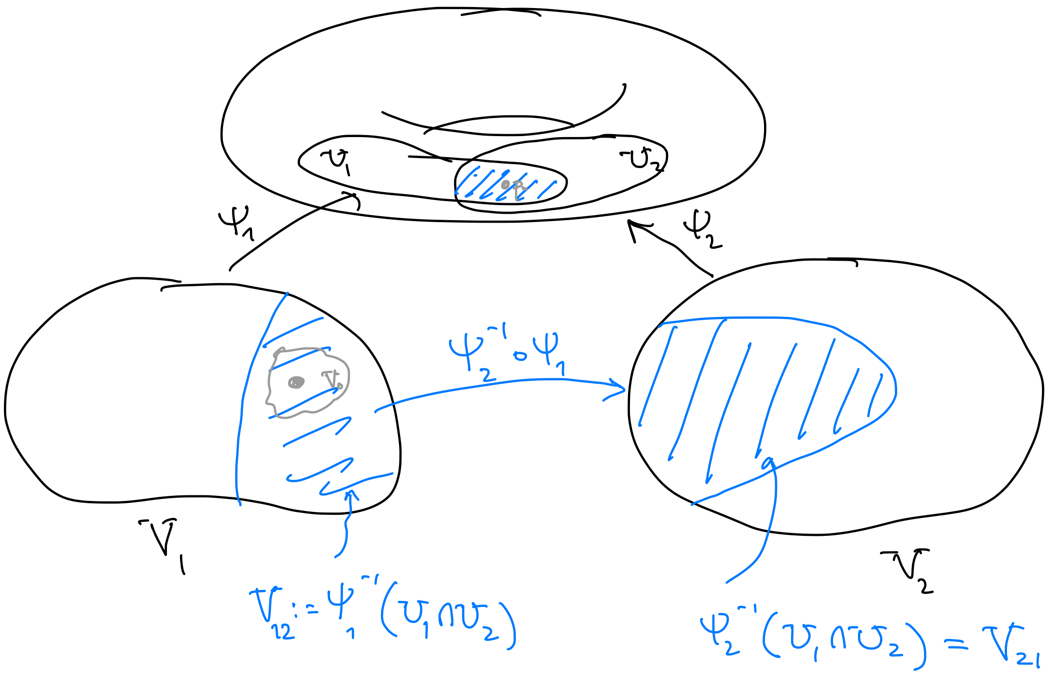
Thus, S is <u>not</u> a surface.

<u>Remark</u> Neither parametrizations, nor local functions as in the Proposition on P. 11 are unique. Our goal is to understand a relation between different parametrizations.

Thus, let $\varphi_1 : V_1 \longrightarrow U_1 \subset S$

$\varphi_2 : V_2 \longrightarrow U_2 \subset S$

be two parametrizations s.t. $U_1 \cap U_2 \neq 0$



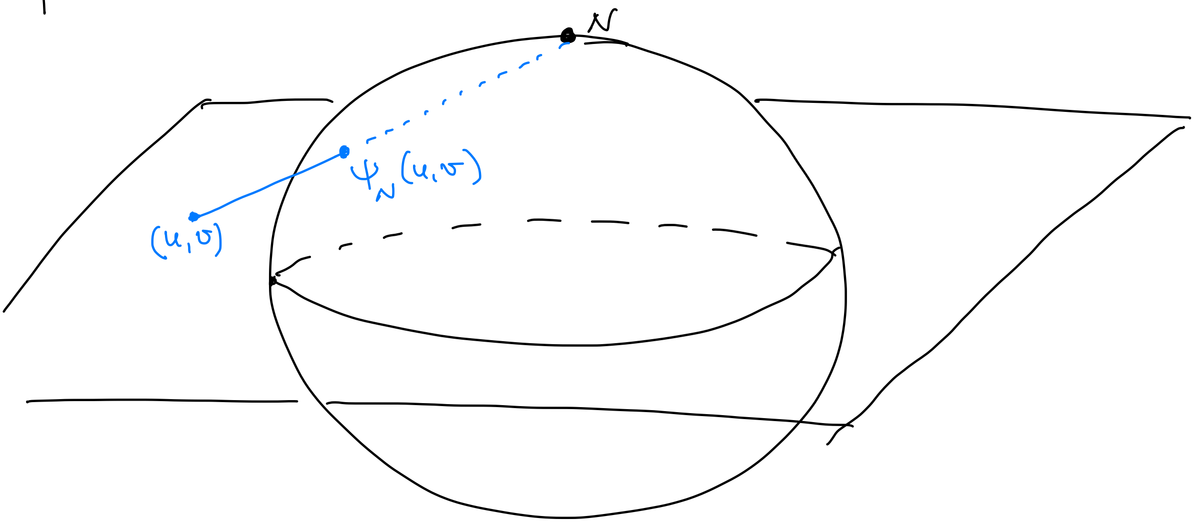
Since φ_1 & φ_2 are homeomorphisms, we have a well-defined continuous map

$$\varphi_{21} := \varphi_2^{-1} \circ \varphi_1 : V_{12} \longrightarrow V_{21}$$

which is called "transition map" or "change of coordinates map".

Notice that φ_{21} is a map $\mathbb{R}^2 \to \mathbb{R}^2$ defined on an open subset.

<u>Ex</u> Consider the sphere S^2, which can be covered by the images of two parametrizations as follows.



The inverse of the steregraphic projection from the north pole N is given by

$$(u,v) \longmapsto \varphi_N(u,v) = \frac{1}{1+u^2+v^2}\left(2u,\ 2v,\ -1+u^2+v^2\right)$$

This is a homeomorphism viewed as a map $\mathbb{R}^2 \to S^2\setminus\{N\}$ and is clearly smooth.

<u>Exercise</u> Show that $D\varphi_N$ is injective at each point.

Thus φ_N is a parametrization (at each point $p\in S^2\setminus\{N\}$).

Of course, we have also the inverse ψ_S of the stereographic projection from the south pole S. The images of these two parametrizations cover together the whole sphere S^2.

Rem A computation shows that the change of coordinates map $\psi_{SN} := \psi_S^{-1} \circ \psi_N : \mathbb{R}^2 \setminus \{0\} \longrightarrow \mathbb{R}^2 \setminus \{0\}$ is given by

$$\psi_{SN}(u, v) = \frac{1}{u^2 + v^2}(u, v)$$

Exercise Show that the sphere can not be covered by the image of a single parametr.

Thm The change of coordinates map is smooth.

Proof Since smoothness is a local property, it suffices to show that $\forall (u_0, v_0) \in V_{12}$ $\exists$ a nbhd $V_0 \subset V_{12}$ such that $\psi_{21}|_{V_0}$ is smooth.

Thus, set $p_0 := \psi_1(u_0, v_0)$. For this p_0 and ψ_2 construct a smooth map $\Phi_2 : W \longrightarrow V_2 \times \mathbb{R}$ as in the proof of the Proposition on P. 11.

Recall that

$$\Phi_2\big|_{S\cap W} : S\cap W \longrightarrow V_2\times\{0\} = V_2$$

equals φ_2^{-1}.

The map $\Phi_2\circ\varphi_1 : \varphi_1^{-1}(S\cap W)\to V_2$ is clearly smooth as a composition of smooth maps. Set $V_0 := V_{12}\cap\varphi_1^{-1}(S\cap W)$.

Since the image of φ_1 lies in S, we have

$$\Phi_2\circ\varphi_1\big|_{V_0} = \varphi_2^{-1}\circ\varphi_1\big|_{V_0} = \varphi_{21}\big|_{V_0}$$

is smooth. ▨

<u>Def</u> Let S be a surface. A function $f : S\to\mathbb{R}$ is said to be smooth, if for any parametrization $\varphi : V\longrightarrow U$ the composition

$$F := f\circ\varphi : V\to\mathbb{R}$$

is smooth. The function $F := f\circ\varphi$ is called a local (coordinate) representation of f.

<u>Rem</u> The theorem on P. 15 implies that if $f \circ \varphi_1$ is smooth, then $f \circ \varphi_2$ is also smooth on $V_{21} = \varphi_2^{-1}(U_1 \cap U_2)$.
Indeed,

$$f \circ \varphi_2 = f \circ \varphi_1 \circ (\varphi_1^{-1} \circ \varphi_2) = \underbrace{(f \circ \varphi_1)}_{\text{smooth}} \circ \underbrace{\varphi_{12}}_{\text{smooth}}$$

Hence, if (V_i, φ_i) is a collection of parametrizations such that $\varphi_i(V_i)$ covers all of S, it suffices to check that $f \circ \varphi_i$ is smooth $\forall i$.

<u>Ex</u> Let $h: \mathbb{R}^3 \longrightarrow \mathbb{R}$ be an arbitrary smooth function. Define $f: S \to \mathbb{R}$ as the restriction of h. Then f is smooth, since for any parametrization φ we have

$$f \circ \varphi = \underbrace{h \circ \varphi}_{\text{smooth.}}.$$

For example, for any fixed $a \in \mathbb{R}^3$ the height function

$$f_a(x) = \langle a, x \rangle \qquad x \in S$$

is a smooth function on S.

In particular, set $S = S^2$ and $h(x,y,z) = z$. Then the coordinate representation of $f = h|_{S^2}$ with respect to ψ_N is

$$F(u,v) = f \circ \psi_N(u,v) = \frac{-1+u^2+v^2}{1+u^2+v^2}.$$

<u>Ex</u> Let $\psi : V \to U$ be a parametrization of a surface S. Since ψ is a homeomorphism, we have the inverse map

$$\varphi := \psi^{-1} : U \to V.$$

Since U itself is a surface (with a single parametrization ψ), it makes sense to ask if φ viewed as a map $U \to \mathbb{R}^2$ is smooth, which means by definition that both components of φ are smooth functions.

This is the case indeed, since the local representation of φ is nothing else but

$$\varphi \circ \psi = id,$$

which is certainly smooth.

Any pair (U, φ) is called <u>a chart</u> on S.

<u>Prop 1</u> Let S be a surface. Then the set $C^\infty(S)$ of all smooth functions on S is a vector space, that is

$$\left.\begin{matrix} f, g \in C^\infty(S) \\ \lambda, \mu \in \mathbb{R} \end{matrix}\right\} \Longrightarrow \lambda f + \mu g \in C^\infty(S).$$

In fact, we also have

$$f, g \in C^\infty(S) \Longrightarrow f\cdot g \in C^\infty(S)$$

<u>Proof</u> We prove the last statement only.
Let $\psi : U \to V$ be a parametrization.
Then

$$(f\cdot g)\circ\psi = \underbrace{\underbrace{(f\circ\psi)}_{\in\, C^\infty(V)}\cdot\underbrace{(g\circ\psi)}_{\in\, C^\infty(V)}}_{\in\, C^\infty(V)}.$$

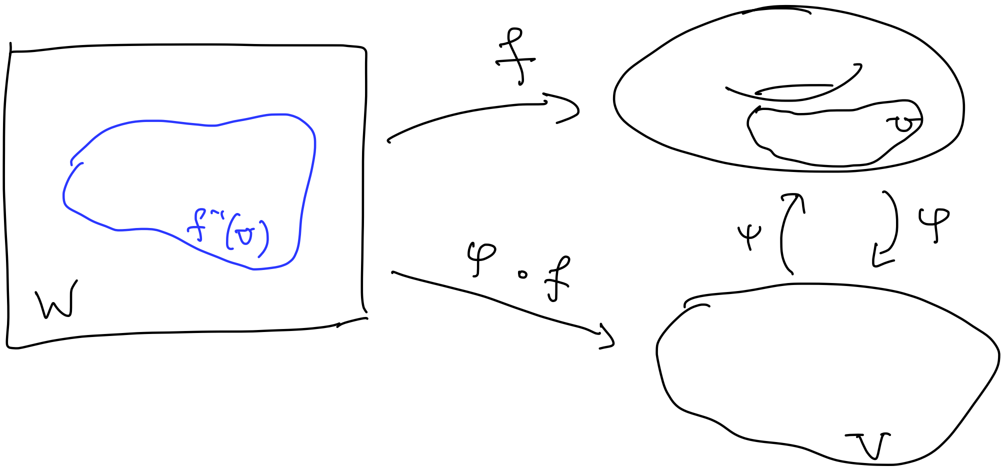
$\boxed{\text{///}}$

Let $W \subset \mathbb{R}^n$ be an open set.

<u>Def</u> A cont. map $f: W \longrightarrow S$, where S is a surface, is called <u>smooth</u>, if for any parametrization $\psi: V \to U \subset S$ the map

$$\varphi\circ f = \psi^{-1}\circ f : \quad f^{-1}(U) \longrightarrow V \subset \mathbb{R}^2$$

is smooth.



<u>Prop</u> $f : W \to S$ is smooth if and only if f is smooth as a map $W \to \mathbb{R}^3$.

More formally, this means the following: If $\imath : S \to \mathbb{R}^3$ denotes the natural inclusion map, then

$$f \in C^\infty(W; S) \iff \imath \circ f \in C^\infty(W; \mathbb{R}^3)$$

<u>Proof</u> Pick a parametrization ψ of S and construct a smooth map $\Phi : X \to \mathbb{R}^3$ just as in the proof of the proposition on P. 11, where $X \subset \mathbb{R}^3$ is an open set.

Assume $f : W \to \mathbb{R}^3$ is smooth. Then $\Phi \circ f$ is also smooth as the composition of smooth maps. However, since f takes values in S and $\Phi|_S = \varphi = \psi^{-1}$, we obtain that $\varphi \circ f = \Phi \circ f : \mathbb{R}^2 \to \mathbb{R}^2$ is smooth.

Conversely, assume that $f: W \to S$ is smooth. Then

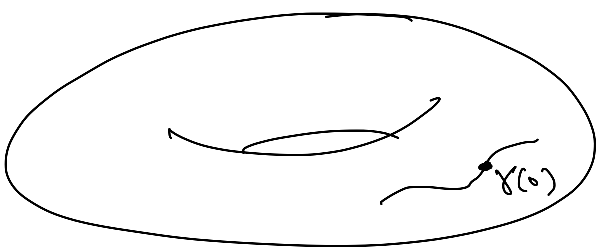
$$f\big|_{f^{-1}(U)} = (\psi \circ \varphi) \circ f\big|_{f^{-1}(U)} = \psi \circ (\varphi \circ f)\big|_{f^{-1}(U)}$$

is again smooth as the composition of smooth maps. ◻

The following class of maps will be particularly important in the sequel.

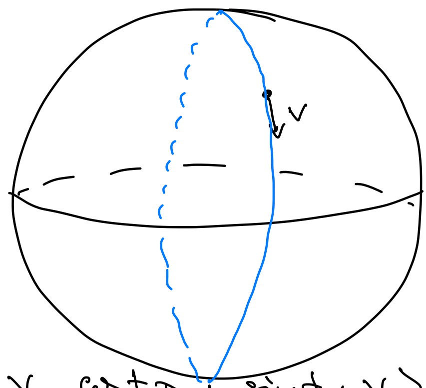
<u>Def</u> Let $I \subset \mathbb{R}$ be an (open) interval. A smooth map $\gamma: I \to S$ is called a smooth curve on S.

If $0 \in I$, we say that γ is a smooth curve through $p := \gamma(0) \in S$.



Ex Let $p \in S^2$ and $v \in \mathbb{R}^3$ s.t. $\langle p, v \rangle = 0$ and $\|v\| = 1$.

Define $\gamma_v : \mathbb{R} \to \mathbb{R}^3$ by



$$\gamma_v(t) = (\cos t) \cdot p + (\sin t) \cdot v.$$

Since

$$\|\gamma_v(t)\|^2 = \langle \cos t \cdot p + \sin t\, v,\ \cos t\, p + \sin t \cdot v \rangle$$

$$= \cos^2 t \cdot \|p\|^2 + 0 + \sin^2 t \cdot \|v\|^2$$

$$= \cos^2 t + \sin^2 t = 1,$$

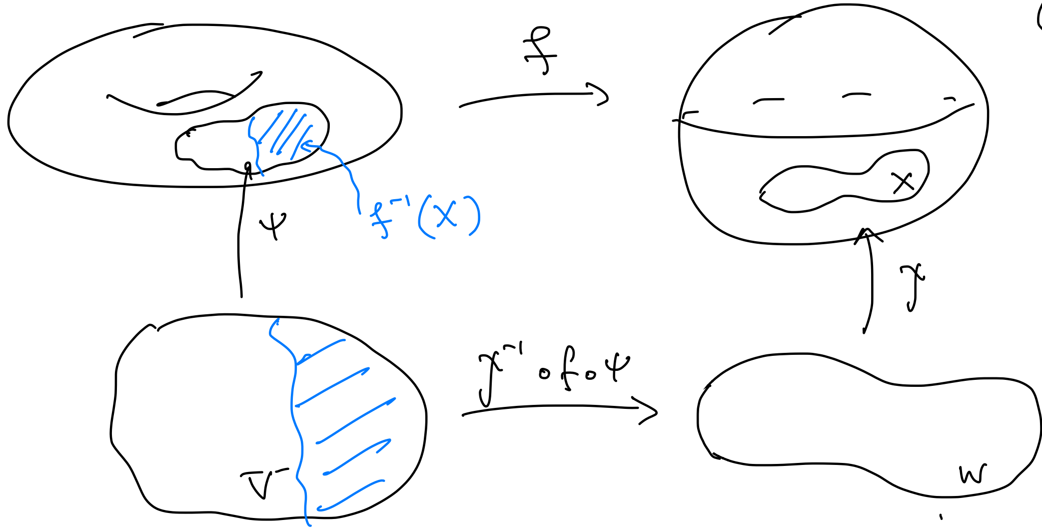
we obtain that $\gamma_v : \mathbb{R} \to S^2$ is a smooth curve through p. Of course, the image of γ_v is a great circle on S^2.

Even more generally, we can define smooth maps between surfaces as follows.

Def Let S_1 and S_2 be two surfaces. A contin. map $f : S_1 \to S_2$ is said to be smooth, if for any parametrizations $\varphi : V \to U \subset S_1$ and $\chi : W \to X \subset S_2$ the map

$$\chi^{-1} \circ f \circ \varphi : \varphi^{-1}(f^{-1}(X)) \longrightarrow W$$

is smooth.



The map $\chi^{-1} \circ f \circ \varphi$ is called the coordinate (or local) representation of f.

<u>Rem</u> Since parametrizations and charts contain the same ammount of information, we can also define smoothness of a map $f : S_1 \to S_2$ in terms of charts as follows: f is smooth if and only if for any chart (V, φ) on S_1 and any chart (X, ξ) on S_2 the map

$$\xi \circ f \circ \varphi^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$$

is smooth (on an open subset where defined)

The map $\xi \circ f \circ \varphi^{-1}$ is also called a coordinate representation of f (with respect to charts (V, φ) and (X, ξ)).

Rem Just like in the case of functions, it suffices to find two collections $\{\varphi_i : V_i \to U_i\}$ and $\{\gamma_j : W_j \to X_j\}$ of parametrizations such that

$$\bigcup_i U_i = S_1 \quad \text{and} \quad \bigcup_j X_j = S_2$$

and check that all coordinate representations $\gamma_j^{-1} \circ f \circ \varphi_i$ are smooth.

Consider the antipodal map

$$a : S^2 \to S^2 , \qquad a(x) = -x .$$

For any $(u,v) \in \mathbb{R}^2$ we have

$$a \circ \varphi_N(u,v) = -\frac{1}{1+u^2+v^2}\left(2u,\ 2v,\ -1+u^2+v^2\right)$$

Since $\varphi_S^{-1} : S^2 \setminus \{s\} \to \mathbb{R}^2$ is given by

$$(x,y,z) \longmapsto \left(\frac{x}{1+z},\ \frac{y}{1+z}\right),$$

we obtain

$$\varphi_S^{-1} \circ a \circ \varphi_N(u,v) = \frac{1}{1+\frac{1-u^2-v^2}{1+u^2+v^2}}\left(-\frac{2u}{1+u^2+v^2},\ -\frac{2v}{1+u^2+v^2}\right)$$

$$= -\frac{1+u^2+v^2}{2}\left(\frac{2u}{1+u^2+v^2},\ \frac{-2v}{1+u^2+v^2}\right)$$

$$= -(u,v)$$

It follows in a similar manner, that

$$\varphi_S^{-1} \circ a \circ \varphi_S , \quad \varphi_N^{-1} \circ a \circ \varphi_N , \text{ and } \varphi_N^{-1} \circ a \circ \varphi_S$$

are also smooth. Hence, a is smooth.

<u>Prop</u> Let $h : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth map such that $h(S_1) \subset S_2$, where S_1 and S_2 are surfaces. Then $h|_{S_1} : S_1 \to S_2$ is also smooth.

The proof of this proposition is similar to the proof of Prop 2 on P. 18 and is left as an exercise to the reader.

To construct a more interesting example, pick a polynomial

$$p(z) := z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$$

with complex coefficients. Identifying $\mathbb{R}^2$ with $\mathbb{C}$, we can view p as a smooth map $\mathbb{R}^2 \to \mathbb{R}^2$.

Define $f: S^2 \to S^2$ by

$$f(p) = \begin{cases} \varphi_N \circ p \circ \varphi_N^{-1}(p) & \text{if } p \neq N, \\ N & \text{if } p = N. \end{cases}$$

I claim that f is smooth. Indeed, since by the construction of f, the coordinate representation of f with respect to the pair $(\mathbb{R}^2, \varphi_N)$ & $(\mathbb{R}^2, \varphi_N)$ of parametrizations (the first one on the source of f, the second one on the target), is

$$\varphi_N^{-1} \circ f \circ \varphi_N = \underbrace{\varphi_N^{-1} \circ \varphi_N}_{id} \circ p \circ \underbrace{\varphi_N^{-1} \circ \varphi_N}_{id} = p.$$

Hence f is smooth at each point $p \in S^2 \setminus \{N\}$.
To check that f is also smooth at $p = N$,

consider

$$\varphi_S \circ f \circ \varphi_S^{-1}(z) = \begin{cases} \varphi_S \circ \varphi_N^{-1} \circ p \circ \varphi_N \circ \varphi_S^{-1}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

We know that

$$\varphi_{SN}(z) = \varphi_S \circ \varphi_N^{-1}(z) = \frac{1}{|z|^2} z = \frac{1}{z \cdot \bar{z}} \cdot z = \frac{1}{\bar{z}}$$

$$\Rightarrow \varphi_{NS}(z) = \varphi_{SN}^{-1}(z) = \frac{1}{\bar{z}}$$

Hence, we compute

$$\varphi_{SN} \circ p \circ \varphi_{NS}(z) = \varphi_{SN}\left(\frac{1}{\bar{z}^n} + \frac{a_{n-1}}{\bar{z}^{n-1}} + \dots + a_0\right)$$

$$= \varphi_{SN}\left(\frac{1 + a_{n-1}\bar{z} + \dots + a_0 \bar{z}^n}{\bar{z}^n}\right)$$

$$= \frac{z^n}{1 + \bar{a}_{n-1} z + \dots + \bar{a}_0 z^n}, \quad z \neq 0.$$

This yields that $\varphi_S \circ f \circ \varphi_S^{-1}$ is smooth even at $z = 0$, that is f is smooth everywhere on S^1 (or, simply, f is smooth).

Thm Suppose $f : S_1 \to S_2$ and $g : S_2 \to S_3$ are smooth maps between surfaces. Then

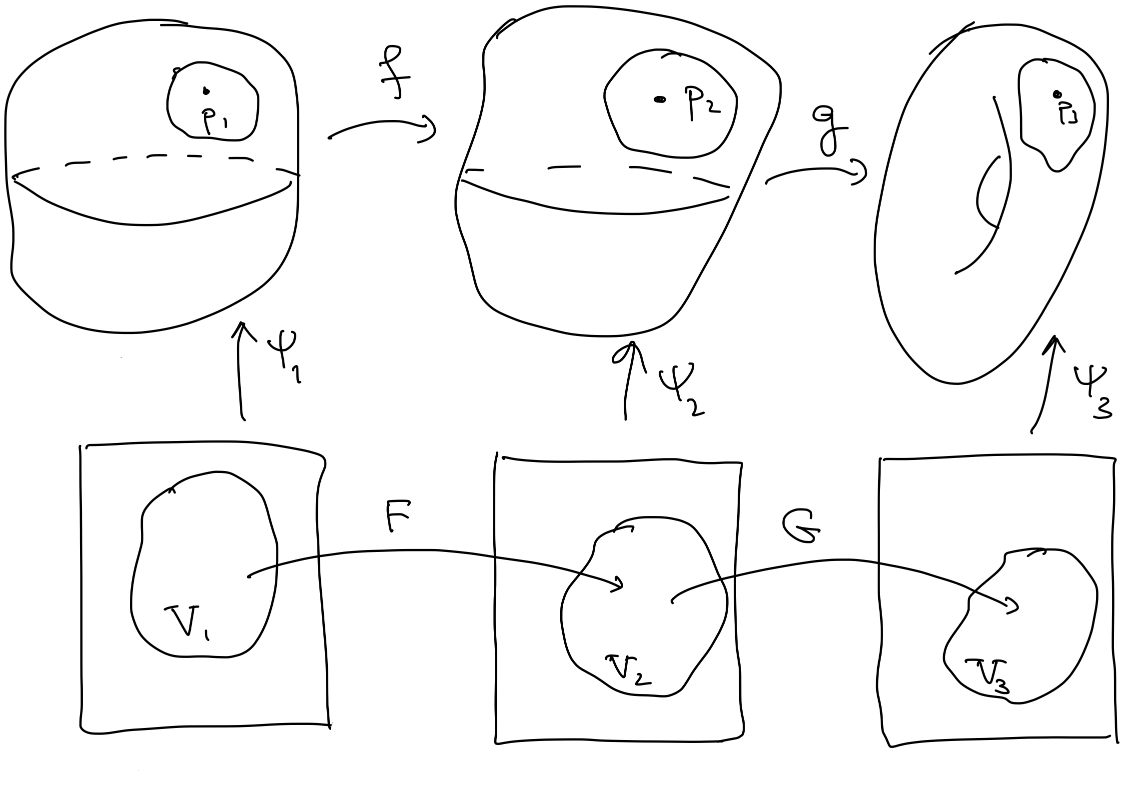
$g \circ f : S_1 \to S_3$ is also smooth.

Proof Pick a pt $p_1 \in S_1$ and denote

$p_2 := f(p_1) \in S_2$, $p_3 := g(p_2) = g(f(p_1)) \in S_3$

Pick parametrizations

$\varphi_j : V_j \to U_j \subset S_j$



In a sufficiently small nbhd of p_1 we have

$$\varphi_3^{-1} \circ (g \circ f) \circ \varphi_1 = \underbrace{\varphi_3^{-1} \circ g \circ \varphi_2}_{G \in C^\infty} \circ \underbrace{\varphi_2^{-1} \circ f \circ \varphi_1}_{F \in C^\infty}$$

$\Rightarrow$ $g \circ f$ is smooth in a nbhd of p_1

$\Rightarrow$ $g \circ f \in C^\infty(S_1 ; S_3)$ ▯

<u>Rem 1</u> The proof shows that the coordinate representation of the composition is the composition of coordinate representations.

<u>Rem 2</u> The proof also shows that the following holds:

If $\gamma : I \to S_1$ is a smooth curve and $f : S_1 \to S_2$ is a smooth map, then $f \circ \gamma : I \to S_2$ is also a smooth curve.

<u>Def</u> A smooth map $f : S_1 \to S_2$ is called a diffeomorphism, if there exists a smooth map $g : S_2 \to S_1$ s.t.

$$g \circ f = id_{S_1} \quad \text{and} \quad f \circ g = id_{S_2}$$

Ex The antipodal map $a: S^2 \to S^2$ is a diffeomorphism.

Ex The hyperboloid $H = \{x^2+y^2-z^2=1\}$ and cylinder $C = \{x^2+y^2=1\}$ are diffeomorphic, that is there exists a diffeomorphism $f: H \to C$.

Explicitly, define

$h: \mathbb{R}^3 \to \mathbb{R}^3$ by $h(x,y,z) = \left(\frac{x}{\sqrt{1+z^2}}, \frac{y}{\sqrt{1+z^2}}, z\right)$

Clearly, $h \in C^\infty(\mathbb{R}^3;\mathbb{R}^3)$. If $(x,y,z) \in H$, then

$$\left(\frac{x}{\sqrt{1+z^2}}\right)^2 + \left(\frac{y}{\sqrt{1+z^2}}\right)^2 = \frac{x^2+y^2}{1+z^2} = 1,$$

that is $f := h|_H : H \longrightarrow C$ is smooth.

Exercise Show that the restriction of $h^{-1}: \mathbb{R}^3 \to \mathbb{R}^3$

$$h^{-1}(u,v,w) = \left(\sqrt{1+w^2}\,u, \sqrt{1+w^2}\,v, w\right)$$

yields a smooth inverse of f.

Rem A map $f: S_1 \to S_2$ may fail to be a diffeomorphism in the following two ways: either f^{-1} does not exist or f^{-1} exists but is not smooth.

<u>Non-example</u> Consider a map

$$f: C \longrightarrow C, \quad f(x,y,z) = (x,y,z^3),$$

which is smooth.

The inverse $f^{-1}: C \to C$ exists:

$$f^{-1}(x,y,z) = (x,y,\sqrt[3]{z})$$

It is continuous, but fails to be smooth.

<u>Exercise</u> Compute a coordinate representation of f^{-1} and check that this fails to be smooth.