The tangent plane

Let S be a surface and $p \in S$.

Def A vector $v \in \mathbb{R}^3$ is said to be tangent to S at p, if $\exists$ a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to S$ s.t.

$$\gamma(0) = p \quad \text{and} \quad \dot{\gamma}(0) = v.$$

When computing the tangent vector of γ we think of γ as a curve in $\mathbb{R}^3$.

Ex $S = S^2 \ni p$ arbitrary. Recall the curve

$$\gamma_v(t) = \cos t \cdot p + \sin t \cdot v,$$

where $\|v\| = 1$ and $v \perp p$. Then

$\dot{\gamma}_v(0) = v$. Hence, v is tangent to S^2 at p.

$T_p S$ = the set of all tangent vectors to S at p.

Prop Let $\psi : V \to U$ be a parametrization such that $\psi(u_0, v_0) = p$. Then

$$T_p S = \operatorname{Im} D_{(u_0, v_0)} \psi.$$

In particular, $T_p S$ is a vector space of dim. 2.

Proof

Step 1 $\quad \operatorname{Im} D_{(u_0,v_0)}\Psi \subset T_pS$

Assume $v \in \operatorname{Im} D_{(u_0,v_0)}\Psi \implies$

$\exists\ w \in \mathbb{R}^2$ s.t. $D_{(u_0,v_0)}\Psi(w) = v$

Consider the $\underbrace{\text{curve}}_{}$ smooth $\beta : (-\varepsilon,\varepsilon) \to V$

$\beta(t) = (u_0, v_0) + t\cdot w.$

Then $\gamma(t) := \Psi\circ\beta(t)$ is a smooth curve in S s.t.

$$\gamma(0) = \Psi(\beta(0)) = \Psi(u_0,v_0) = p,$$

$$\dot\gamma(0) = D_{(u_0,v_0)}\Psi(w) = v.$$

$\implies v \in T_pS.$

Step 2 $\quad T_pS \subset \operatorname{Im} D_{(u_0,v_0)}\Psi$

If $v \in T_pS$, then $\exists\ \gamma : (-\varepsilon,\varepsilon) \to S$ s.t. $\gamma(0) = p$ & $\dot\gamma(0) = v$. Can assume $\operatorname{Im}\gamma \subset U$ by choosing ε smaller if necessary.

If $\varphi = \Psi^{-1}$, then

$\beta(t) := \varphi\circ\gamma(t)$ is a smooth curve in $V \subset \mathbb{R}^2$ s.t. $\beta(0) = (u_0, v_0)$.

Denote $w := \dot\beta(0) \in \mathbb{R}^2$. Then we have

$$v = \dot\gamma(0) = (\underbrace{\Psi \circ \beta}_{\Psi \circ \Psi^{-1} \circ \gamma})^{\bullet}(0) = (D_{(u_0,v_0)}\Psi)(\dot\beta(0))$$

$$= D_{(u_0,v_0)}\Psi(w) \in \operatorname{Im} D_{(u_0,v_0)}\Psi.$$

<u>Step 3</u> $\dim T_pS = 2$.

This follows immediately from the injectivity of $D_{(u_0,v_0)}\Psi$.

$\blacksquare$

<u>Exercise</u> Assume $v \perp p \in S^2$ and $\|v\| = 2$. Find a curve in S^2 through p with the tangent vector v.

<u>Prop</u> Pick $p \in S$ and recall that there exists a nbhd $W \subset \mathbb{R}^3$ of p and a smooth function $\varphi : W \to \mathbb{R}$ s.t.

$$S \cap W = \{ q \in W \mid \varphi(q) = 0 \} \text{ and } \nabla\varphi(q) \neq 0 \quad \forall q \in W.$$

Then

$$T_pS = \nabla\varphi(p)^{\perp}.$$

Proof If γ is any curve in S through p, then

$$\varphi \circ \gamma(t) = 0 \quad \forall t \Rightarrow \underbrace{\frac{d}{dt}\Big|_{t=0} \varphi(\gamma(t))}_{= \langle \nabla\varphi(p), \dot\gamma(0)\rangle} = 0$$

$$\Rightarrow \underbrace{T_pS \subset \nabla\varphi(p)^\perp}_{\text{both have dimension } 2} \Rightarrow T_pS = \nabla\varphi(p)^\perp$$

$\blacksquare$

Ex 1) $\forall p \in S^2 \quad T_pS^2 = p^\perp$

Indeed, for $\varphi(x,y,z) = x^2+y^2+z^2-1$ we have $\nabla\varphi(x,y,z) = 2(x,y,z)$.

2) $p = (x,y,z) \in H = \{\underbrace{x^2+y^2-z^2-1}_{\varphi(x,y,z)} = 0\}$

$$\nabla\varphi(p) = 2(x,y,-z) \neq 0$$

$$\Rightarrow T_pH = (x,y,-z)^\perp$$

$$= \{ v = (v_1, v_2, v_3) \in \mathbb{R}^3 \mid xv_1 + yv_2 - zv_3 = 0 \}.$$

3) $p = (x,y,z) \in C = \{x^2+y^2 = 1\}$

$$T_pC = \{ v = (v_1,v_2,v_3) \mid xv_1 + yv_2 = 0, \ v_3 \text{ arbitrary} \}.$$

Differential of a smooth map

Let S be a surface and $f \in C^\infty(S)$.
Define a map $d_p f : T_p S \longrightarrow \mathbb{R}$ as follows:

for $v \in T_p S$ choose a smooth curve γ throught p with $\dot\gamma(0) = v$ and set

$$d_p f(v) = \frac{d}{dt}\Big|_{t=0} f \circ \gamma(t).$$

<u>Prop</u> $d_p f$ is a well-defined linear map.

<u>Proof</u>

<u>Step 1</u> $d_p f$ is well-defined.

If γ_1 and γ_2 are two curves through p s.t. $\dot\gamma_1(0) = v = \dot\gamma_2(0)$, then for $\beta_j := \psi^{-1} \circ \gamma_j$ we have

$$\gamma_j(t) = \psi \circ \beta_j(t) \Rightarrow v = D_0\psi(\dot\beta_1(0)) = D_0\psi(\dot\beta_2(0))$$

injective

$$\Rightarrow \dot\beta_1(0) = \dot\beta_2(0) =: w$$

$$\frac{d}{dt}\Big|_{t=0} f \circ \gamma_1(t) = \frac{d}{dt}\Big|_{t=0} \left(f \circ \psi \circ \psi^{-1} \circ \gamma_1(t) \right)$$

$$= \frac{d}{dt}\left(F \circ \beta_1(t) \right)$$

$$= D_0 F(w)$$

Similarly, $\frac{d}{dt}\Big|_{t=0} f\circ\gamma_2(t) = D_0F(w)$

$$\Rightarrow \frac{d}{dt}\Big|_{t=0}\left(f\circ\gamma_1(t)\right) = \frac{d}{dt}\Big|_{t=0}\left(f\circ\gamma_2(t)\right).$$

<u>Step 2</u> $\quad d_pf\circ D_0\Psi = D_0F$, where $F := f\circ\Psi$.

This follows from the pf of Step 1.

<u>Step 3</u> $\quad d_pf$ is linear

$$\underbrace{d_pf\circ D_0\Psi = D_0F}_{\text{linear}} \quad\Rightarrow\quad d_pf \text{ is linear}$$

$\boxed{\text{Q.E.D.}}$

<u>Exercise</u> If $h\in C^\infty(\mathbb{R}^3)$ and $f = h|_S$, then $\forall p\in S$ we have

$$d_pf = D_ph\big|_{T_pS}.$$

<u>Def</u> A point $p\in S$ is called critical for $f\in C^\infty(S)$, if $d_pf = 0$, that is $d_pf(v)=0$ $\forall v\in T_pS$.

<u>Prop</u> If p is a pt of loc. max. (min) for f, then p is critical for f.

<u>Proof</u> p is a pt of loc. max for f $\Rightarrow$

$\forall$ curve γ through p, 0 is a pt of loc. max for $f\circ\gamma$

$\Rightarrow \frac{d}{dt}\Big|_{t=0} f\circ\gamma = 0$ $\boxed{\text{Q.E.D.}}$

Prop Let $h, \varphi \in C^\infty(\mathbb{R}^3)$. Assume $\nabla\varphi(p) \neq 0$ for any $p \in S = \varphi^{-1}(0)$. If $p \in S$ is a pt of loc. max. for $f = h|_S$, then

$$\nabla h(p) = \lambda \nabla\varphi(p)$$

for some $\lambda \in \mathbb{R}$.

Proof Notice: S is a surface and

$$T_pS = \{ v \in \mathbb{R}^3 \mid \langle v, \nabla\varphi(p)\rangle = 0 \}$$

$$d_pf = 0 \iff D_ph|_{T_pS} = 0$$

$$\iff \langle v, \nabla h(p)\rangle = 0 \quad \forall v \in T_pS$$

$$\iff \nabla h(p) = \lambda\nabla\varphi(p) \text{ for some } \lambda \in \mathbb{R} \qquad \blacksquare$$

Rem This proof is in a sense more conceptual then the pf of Thm 1 on P. 4 of Part 1.

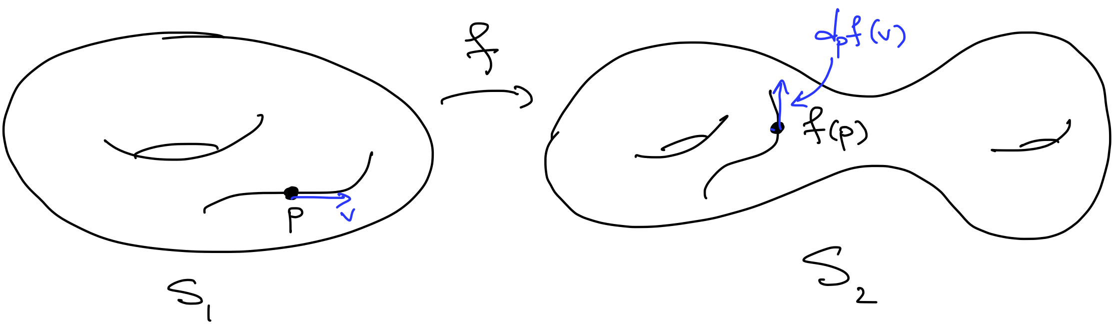
More generally, for any $f \in C^\infty(S; \mathbb{R}^n)$ the differential $d_pf : T_pS \to \mathbb{R}^n$ is defined by the same formula.

Also, the differential is well-defined for maps $f : \mathbb{R}^n \to S$, $d_pf : \mathbb{R}^n \to T_{f(p)}S$

$$f : S_1 \to S_2, \quad d_pf : T_pS_1 \to T_{f(p)}S_2$$

In the latter case, if $\dot\gamma(0) = v \in T_pS_1$, then

$$d_p f\,(v) = \frac{d}{dt}\Big|_{t=0} \big(f\circ\gamma(t)\big)$$



<u>Prop</u> Let S_1, S_2, S_3 be smooth surfaces. For any smooth maps $f: S_1 \to S_2$ and $g: S_2 \to S_3$ and any pt $p \in S_1$ we have

$$\boxed{D_p(g\circ f) = D_{f(p)}g \circ D_p f.}$$

This also holds if any of S_i is replaces by an open subset of $\mathbb{R}^n$.

<u>Proof</u> Let γ_1 be any smooth curve in S_1 through p. Denote $\gamma_2 = f\circ\gamma$, which is a smooth curve in S_2 through $f(p)$. If $\dot\gamma_1(0) = v_1$, then $v_2 := \dot\gamma_2(0) = D_p f\,(v_1)$ by the definition of $D_p f$. Hence,

$$D_p(g\circ f)(v_1) = \frac{d}{dt}\Big|_{t=0}\big(g\circ \underbrace{f\circ\gamma_1}_{\gamma_2}(t)\big)$$

$$= \frac{d}{dt}\Big|_{t=0}\big(g\circ\gamma_2(t)\big) = D_{f(p)}g\,(v_2)$$

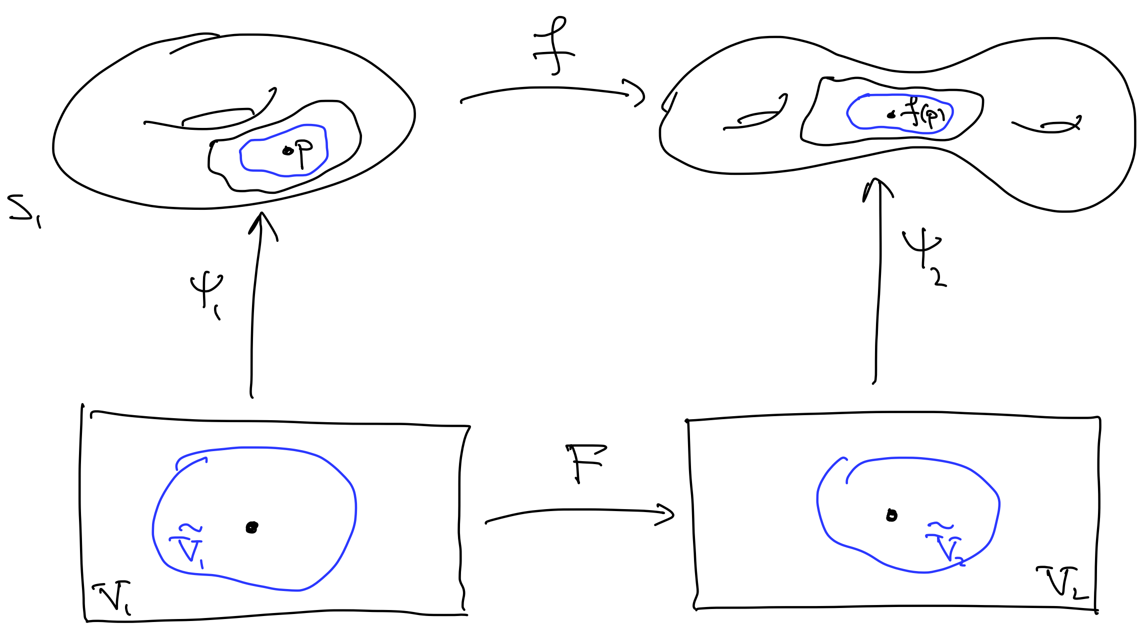
$$= D_{f(p)}g\,(D_p f\,(v_1)).$$

Cor If $f: S_1 \to S_2$ is a diffeomorphism, then $\forall p \in S_1$ $d_pf: T_pS_1 \to T_{f(p)}S_2$ is an isomorphism. $\square$

Def A map $f: S_1 \to S_2$ is called a <u>local diffeomorphism</u> if $\forall p \in S_1$ $\exists$ a nbhd $U_1 \subset S_1$ and a nbhd $U_2 \subset S_2$ of $f(p)$ s.t. $f: U_1 \to U_2$ is a diffeomorphism.

Thm Let $f: S_1 \to S_2$ be a smooth map s.t. $\forall p \in S_1$ $d_pf: T_pS_1 \to T_{f(p)}S_2$ is an isomorphism. Then f is a local diffeomorphism.

Proof Pick any $p \in S_1$ and parametrizations $\varphi_1 : V_1 \to W_1 \subset S_1$ and $\varphi_2 : V_2 \subset W_2 \subset S_2$



Without loss of generality $\varphi_1(0) = p$ and $\varphi_2(0) = f(p)$.

$$F = \varphi_2^{-1} \circ f \circ \varphi_1 \implies d_0 F = d_{f(p)} \varphi_2^{-1} \circ d_p f \circ d_0 \varphi_1$$

$d_0 \varphi_1 : \mathbb{R}^2 \to T_p S_1$ is an iso.

$d_{f(p)} \varphi_2 : T_{f(p)} S_2 \to \mathbb{R}^2$ is an iso.

$d_p f$ is an iso $\implies d_0 F$ is an iso.

From analysis, it is known that $\exists$ a nbhd $0 \in \tilde{V}_1 \subset V_1$ and a nbhd $\tilde{V}_2 \subset V_2$ of 0 such that $F : \tilde{V}_1 \to \tilde{V}_2$ is a diffeomorphism

Denote $U_1 = \varphi_1(\tilde{V}_1)$, $U_2 = \varphi_2(\tilde{V}_2)$.

Then

$$f\big|_{U_1} = \varphi_2 \circ F \circ \varphi_1^{-1}\big|_{U_1} : U_1 \to U_2$$

is a diffeomorphism, since it is a composition of diffeomorphisms. ▨

<u>Rem</u> It follows from the proof, that

$$d_p f = \underbrace{d_0 \varphi_2 \circ d_0 F \circ d_p \varphi^{-1}}_{\text{iso}}.$$

In particular,

$d_p f$ is injective $\iff$ $D_{\varphi_1(p)}$ is injective

surj. $\iff$ surj.

iso $\iff$ iso

Let $f \in C^\infty(S_1; S_2)$.

Def 1) A pt $p \in S_1$ is called a critical pt of f if $d_p f$ is not surjective $\iff$ $d_p f$ is not injective $\iff$ $d_p f$ is not an iso.

2) $q \in S_2$ is called a regular value of f, if $\forall p \in f^{-1}(q)$ is regular (non-critical), i.e., if $\forall p \in f^{-1}(q)$ $d_p f$ is surjective $\iff$ $d_p f$ is injective $\iff$ $d_p f$ is an iso.

Ex $f : \mathbb{R}^2 \to \mathbb{R}^2 \cong \mathbb{C}$, $f(z) = z^n$, $n \in \mathbb{Z}$, $n \geq 2$. It is known from analysis that $D_z f : \mathbb{C} \to \mathbb{C}$ can be identified with the map $h \longmapsto f'(z) \cdot h$. Hence, z is critical iff $f'(z) = 0 \iff n z^{n-1} = 0$

$\iff z = 0$. Hence, f has a single critical pt $z = 0$ and a single critical value, the zero. All other pts are regular and any non-zero value is also regular.

Thm (The fundamental theorem of algebra)
Let $q(z) = z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$
be a polynomial of degree $n \geq 1$ with cx. coefficients. Then p has at least one cx. root.

Proof Recall that the map $f: S^2 \to S^2$,

$$f(p) = \begin{cases} N & p = N, \\ \varphi_N \circ q \circ \varphi_N^{-1}, & p \neq N, \end{cases}$$

is smooth.

Step 1 f has has at most n crit pts (values).

$p \in S^2 \setminus \{N\}$ is critical $\Longleftrightarrow$ $z := \varphi_N(p)$ is critical for q $\Longleftrightarrow$ $q'(z) = 0$, that is

$$n z^{n-1} + (n-1) a_{n-1} z^{n-2} + \dots + a_1 = 0$$

This can have at most $(n-1)$ roots.

Step 2 Denote by $\mathcal{R}(f)$ the set of regular values of f. Then for any $r \in \mathcal{R}(f)$ the set $f^{-1}(r)$ is finite and the map

$$\mathcal{R}(f) \to \mathbb{Z}_{\geq 0}, \qquad r \longmapsto \# f^{-1}(r)$$

is constant.

Let $p \in f^{-1}(r)$, $r \in \mathcal{R}(f) \Longrightarrow$

$f(p) = r$ & $d_p f$ is an iso

$\Rightarrow \exists$ a nbhd U_p of p and a nbhd W_r s.t.

$f : U_p \longrightarrow W_r$ is a diffeo. In particular, $f^{-1}(r) \cap U_p = \{p\} \Longrightarrow$ $f^{-1}(r)$ is discrete.

However, $f^{-1}(r)$ is a closed subset of S^2, hence compact. But a compact discrete set must be finite.

Denote $f^{-1}(r) = \{p_1, \ldots, p_m\}$ and the corresponding nbhds $U_1, \ldots, U_m$, $W_1, \ldots, W_m$.

Set $W := W_1 \cap \ldots \cap W_m$ and $\tilde{U}_j := f^{-1}(W) \cap U_j$. Then for each $j \leq m$ the map $f : \tilde{U}_j \longrightarrow W$ is a diffeomorphism.

In particular, $\forall\, r' \in W \; \exists!\, p_j' \in \tilde{U}_j$ s.t. $f(p_j') = r'$. Hence, $\# f^{-1}(r') = \# f^{-1}(r)$ $\forall\, r' \in W$, so that the function

$$\mathcal{R}(f) \longrightarrow \mathbb{Z}, \quad r \longmapsto \# f^{-1}(r) \qquad (*)$$

is locally constant.

However, $\mathcal{R}(f)$ is the complement of a finite number of pts in S^2, hence connected.

Therefore, $(*)$ is (globally) constant.

Step 3 We prove this thm.

Pick any pairwise distinct pts $p_1, \dots, p_{n+1} \in S^2 \setminus \{N\}$ s.t. $f(p_1), \dots, f(p_{n+1})$ are also pairwise distinct. Since f has at most n critical values, at leas one pt from $\{f(p_1), \dots, f(p_{n+1})\}$ is a regular value and $(*)$ does not vanish at this pt. Hence, $(*)$ vanishes nowhere on $\mathcal{R}(f)$.

If S is a critical value of S, then $f^{-1}(S) \neq \emptyset$, since $f^{-1}(S)$ contains a critical pt. However,

$$f^{-1}(S) \neq \emptyset \iff g^{-1}(0) \neq 0.$$

If S is a regular value, then by Step 2 $\# f^{-1}(S) \geq 1 \implies f^{-1}(S) \neq \emptyset$.

This finishes the proof. $\blacksquare$

Orientability

Let $S \subset \mathbb{R}^3$ be a (smooth) surface.

<u>Def</u> 1) A (smooth) map $v : S \to \mathbb{R}^3$ is called a (smooth) tangent vector field on S, if $\forall p \in S \quad v(p) \in T_pS$.

2) A (smooth) map $n : S \to \mathbb{R}^3$ is called a (smooth) <u>normal field</u> on S, if $\forall p \in S \quad n(p) \perp T_pS$.

<u>Ex</u> Set $S = S^2$, $n(x) = x$. Then n is a normal vector field on S^2.

<u>Lemma</u> Let $\psi : V \to U \subset S$ be a parametrization. Then U admits a unit normal field n on U, that is $\forall p \in U \; n(p) \perp T_pS$ and $|n(p)| = 1$.

<u>Proof</u> ψ is a param. $\Longrightarrow$
$\forall p \in U \quad \exists!\, q \in V$ s.t. $\psi(q) = p$ and

$$D_q\psi : \mathbb{R}^2 \to T_pS = \mathrm{Im}(D_q\psi)$$

is an isomorphism. Hence, $D_q\psi$ maps a basis of $\mathbb{R}^2$ onto a basis of T_pS. Therefore, the image of the standard basis $(\partial_u\psi, \partial_v\psi)|_q$ is a basis of T_pS.

Define

$$n(p) = \frac{\partial_u \varphi \times \partial_v \varphi}{|\partial_u \varphi \times \partial_v \varphi|},$$

where "$\times$" means the cross-product in $\mathbb{R}^3$. This is well-defined, since $\partial_u \varphi \times \partial_v \varphi \neq 0$.

Exercise: Check that n is smooth. ▨

<u>Lemma</u> If S is connected, then there are at most 2 non-equal unit normal fields on S.

<u>Proof</u> Let n_1 and n_2 be unit norm. fields.

$$\forall p \in S \qquad \left.\begin{aligned} &n_1(p) \perp T_pS \\ &n_2(p) \perp T_pS \\ &|n_1(p)| = 1 = |n_2(p)| \end{aligned}\right\} \Rightarrow n_2(p) = \pm n_1(p)$$

$$S_\pm := \{p \in S \mid n_2(p) = \pm n_1(p)\}.$$

Then both S_+ and S_- are closed and $S = S_+ \cup S_-$. Hence, either

$$S_+ = \emptyset \iff n_2(p) = -n_1(p) \quad \forall p \in S$$

or

$$S_- = \emptyset \iff n_2(p) = n_1(p) \quad \forall p \in S.$$ ▨

<u>Def</u> A surface S is said to be <u>orientable</u>, if S admits a unit normal field.

It should be intuitively clear that any unit normal field "selects a side" of the surface. A choice of the unit normal field ("a side of S") is called an orientation of S. Thus, any surface S admits at most 2 distinct orientations.

<u>Prop</u> (Preimages are orientable)
If 0 is a regular value of $\varphi : \mathbb{R}^3 \to \mathbb{R}$, then $S := \varphi^{-1}(0)$ admits a unit normal field.

Here, just like in the definition on P. 11, 0 is said to be the regular value of φ if $\forall p \in S = \varphi^{-1}(0)$
$D_p \varphi : \mathbb{R}^3 \to \mathbb{R}$ is surjective

$\Longleftrightarrow$ $\nabla \varphi(p) \neq 0$, since $D_p \varphi(v) = \langle \nabla\varphi(p), v \rangle$
$v \in \mathbb{R}^3$.

<u>Proof of the proposition</u>

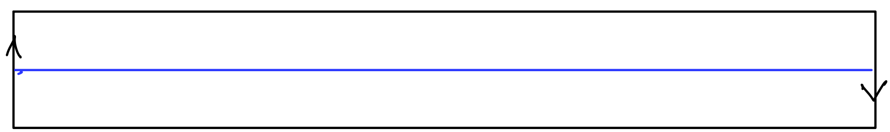
$T_p S = \nabla \varphi(p)^{\perp} \Longrightarrow \nabla\varphi$ is a normal field

0 is a regular value of $\varphi \Longrightarrow \nabla \varphi$ vanishes nowhere

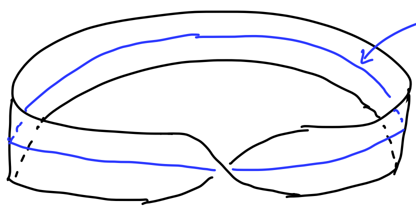
$$\Longrightarrow \quad n(p) := \frac{\nabla\varphi(p)}{|\nabla\varphi(p)|} \quad \text{is a unit normal field} \qquad \blacksquare$$

Rem In the definition of orientability, it is only important, that the normal field exists, is non-vanishing and continuous.

A non-example: the Möbius band.
One can obtain the Möbius band from the strip by gluing the opposite sides as shown on the figure.



gluing



More formally, the Möbius band is the image of the map

$$\Psi : [0, 2\pi] \times (-1,1) \longrightarrow \mathbb{R}^3,$$

$$\Psi(u,v) = \left(\left(2 - v \sin \tfrac{u}{2}\right) \sin u, \left(2 - v \sin \tfrac{u}{2}\right) \cos u, v \cos \tfrac{u}{2} \right)$$

Exercise Show that the image of Ψ is a surface indeed.

We showed that any point on a surface admits an orientable neighbourhood U. Moreover, it follows from the proof that given $0 \neq n_0 \perp T_{p_0}S$ at some $p_0 \in U$, there is a unique orientation n of U such that $n(p_0) = \frac{n_0}{|n_0|}$.

With this understood, $\forall p \in L$ pick an orientable neighbourhood U_p. Since L is compact, there is a finite collection $U_1, \dots, U_n$ covering L. Choose a pt $p_1 \in L \cap U_1$ and a vector $n_1 \in T_{p_1}S^{\perp}$, $|n_1| = 1$. This determines uniquely a normal field n on U_1 such that $n(p_1) = n_1$. If $U_2 \cap U_1 \neq \emptyset$, then there exists a unique smooth extension of n to $U_1 \cup U_2$. After finitely many steps we obtain a normal field n on $U_1 \cup \dots \cup U_n \supset L$. However, as one travels once along L, this normal field must change its direction, that is $n(p_1) = -n(p_1)$, which is impossible. Hence, the Möbious band does not admit a unit normal field, that is the Möbius band is non-orientable.

Let S be a surface.

<u>Def</u> A collection $\mathcal{A} = \{ (\varphi_a, V_a, U_a) \mid a \in A \}$ of parametrizations of S is said to be an atlas, if $\bigcup_{a \in A} U_a = S$.

Recall that for $a, b \in A$ the map

$$\theta_{ab} := \varphi_a^{-1} \circ \varphi_b : \underset{\substack{\cap \\ \mathbb{R}^2}}{V_{ab}} \to \mathbb{R}^2$$

is called the change of coordinates map.

<u>Def</u> An atlas $\mathcal{A}$ on S is said to be oriented, if

$$\det (D_{(u,v)} \theta_{ab}) > 0$$

for any $(u,v) \in V_{ab}$.

<u>Ex</u> For $S = S^2$, $\mathcal{A} = \{ (\varphi_N, \mathbb{R}^2, S^2 \setminus \{N\}), (\varphi_S, \mathbb{R}^2; S^2 \setminus \{S\}) \}$ is an atlas. We have

$$\theta_{SN}(u, v) = \frac{1}{u^2 + v^2} (u, v)$$

A computation yields $\det (D\theta_{SN}) < 0$, so that $\mathcal{A}$ is <u>not</u> an oriented atlas.

Consider, however

$$\mathcal{B} = \{ (\varphi_N, \mathbb{R}^2, S^2 \setminus \{N\}), (\hat{\varphi}_S, \mathbb{R}^2, S^2 \setminus \{S\}) \},$$

where $\hat{\varphi}_S(u, v) = \varphi_S(-u, v) = \varphi_S \circ \sigma(u, v)$, where $\sigma(u,v) = (-u,v)$. Then

$$\hat{\Theta}_{SN} = \hat{\varphi}_S^{-1} \circ \varphi_N = (\varphi_S \circ \sigma)^{-1} \circ \varphi_N = \underset{\sigma}{\overset{\sigma^{-1}}{\shortparallel}} \circ \Theta_{SN}$$

σ is linear $\Rightarrow D\hat{\Theta}_{SN} = \sigma \circ D\Theta_{SN}$

$$\Rightarrow \det D\hat{\Theta}_{SN} = \underset{-1}{\underset{\shortparallel}{\det \sigma}} \cdot \underset{0}{\underset{\wedge}{\det D\Theta_{SN}}} > 0$$

$\Rightarrow$ $\mathcal{B}$ is an oriented atlas on S^2.

<u>Prop</u> A surface S is orientable if an only if S admits an oriented atlas.

<u>Proof</u>

<u>Step 1</u> If S is orientable, then S admits an oriented atlas.

Choose a unit normal field n on S and an atlas $\mathcal{A}$ on S.

Define a new atlas $\mathcal{B}$ as follows:

If $\varphi_a : V_a \to U_a$ belongs to $\mathcal{A}$ and

$\det(\partial_u \varphi_a, \partial_v \varphi_a, n(\varphi_a(u,v))) > 0$, then

(φ_a, V_a, U_a) belongs to $\mathcal{B}$. If $\det(\;) < 0$,

then $(\varphi_a \circ \sigma, \sigma(V_a), U_a) = (\hat{\varphi}_a, \hat{V}_a, U_a)$

belongs to $\mathcal{B}$, where $\sigma : \mathbb{R}^2 \to \mathbb{R}^2$, $\sigma(u,v) = (-u, v)$.

$$\Rightarrow \det(\partial_u \hat{\varphi}_a, \partial_v \hat{\varphi}_a, n(\hat{\varphi}_a(u,v)))$$

$$= \det(-\partial_u \varphi_a, \partial_v \varphi_a, n(\;)) > 0.$$

We obtain:

a) If $\varphi_a, \varphi_b \in \mathcal{B}$, then

$$\underset{(x_1,x_2)}{\varphi_a : V_a} \longrightarrow U_a \quad ; \quad \underset{(y_1,y_2)}{\varphi_b : V_b} \longrightarrow U_b$$

$\theta_{ab} : V_b \longrightarrow V_a$ (defined on a subset of V_b)

$$y \longmapsto \theta(y) = \theta_{ab}(y)$$

$$\varphi_b = \varphi_a \circ \theta \Rightarrow$$

$$\partial_{y_1} \varphi_b = \partial_{x_1} \varphi_a(\theta(y))\, \partial_{y_1} \theta_1 + \partial_{x_2} \varphi_a(\theta(y))\, \partial_{y_1} \theta_2$$

$$\partial_{y_2} \varphi_b = \partial_{x_2} \varphi_a(\theta(y))\, \partial_{y_2} \theta_1 + \partial_{x_2} \varphi_a(\theta(y))\, \partial_{y_2} \theta_2 .$$

Or simply $(\partial_{y_1} \varphi_b, \partial_{y_2} \varphi_b) = (\partial_{x_1} \varphi_a, \partial_{x_2} \varphi_a) \cdot \partial_y \theta$, where

$$\partial_y \Theta = \begin{pmatrix} \partial_{y_1}\Theta_1 & \partial_{y_2}\Theta_1 \\ \partial_{y_1}\Theta_2 & \partial_{y_2}\Theta_2 \end{pmatrix}$$

is the Jacobi matrix of $\Theta = \Theta_{ab}$.

$$\Longrightarrow \left(\partial_{y_1}\varphi_b, \partial_{y_2}\varphi_b, n\right) = \left(\partial_{x_1}\varphi_a, \partial_{x_2}\varphi_a, n\right)\left(\begin{array}{cc|c} \multicolumn{2}{c|}{\partial_y\Theta} & \begin{matrix}0\\0\end{matrix} \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$\Longrightarrow \underbrace{\det\left(\partial_{y_1}\varphi_b, \partial_{y_2}\varphi_b, n\right)}_{>0} =$$

$$= \underbrace{\det\left(\partial_{x_1}\varphi_a, \partial_{x_2}\varphi_a, n\right)}_{>0} \cdot \underbrace{\det\left(\begin{array}{c|c} \partial_y\Theta & \\ \hline & 1 \end{array}\right)}_{=\det(\partial_y\Theta)}$$

by the assumption

$$\Longrightarrow \det(\partial_y\Theta) > 0.$$

b) If $\hat{\varphi}_a$ and $\hat{\varphi}_b$ belong to $\mathcal{B}$, the same computation yields

$$\det(\partial_y\Theta_{ab}) > 0.$$

We have: $\varphi_b = \varphi_a \circ \Theta_{ab} \Longrightarrow$

$$\Longrightarrow \underbrace{\varphi_b \circ \sigma}_{=\hat{\varphi}_b} = \varphi_a \circ \Theta_{ab} \circ \sigma = \underbrace{(\varphi_a \circ \sigma)}_{=\hat{\varphi}_a} \circ \sigma \circ \Theta_{ab} \circ \sigma$$

Hence, the change of coordinates map between

$\hat{\varphi}_a$ and $\hat{\varphi}_b$ is $\hat{\theta}_{ab} := \sigma \circ \theta_{ab} \circ \sigma$

$$\Rightarrow \det(D\hat{\theta}_{ab}) = \det \sigma \cdot \det D\theta_{ab} \cdot \det \sigma$$
$$= (\det \sigma)^2 \det D\theta_{ab} > 0.$$

c) φ_a and $\hat{\varphi}_b$ belong to $\mathcal{B}$

A similar computation yields $\det(D\theta_{ab}) < 0$ and if $\hat{\theta}_{ab}$ is the change of coordinates between φ_a and $\hat{\varphi}_b$, then

$$\hat{\theta}_{ab} = \theta_{ab} \circ \sigma \Rightarrow$$

$$\det D\hat{\theta}_{ab} = \underbrace{\det(D\theta_{ab})}_{<0} \cdot \underbrace{\det \sigma}_{<0} > 0.$$

Thus, $\mathcal{B}$ is an oriented atlas.

<u>Step 2</u> If S admits an oriented atlas, then S admits a unit normal field.

Let $\mathcal{A}$ be an oriented atlas on S and $\varphi_a : V_a \to U_a$ a parametrization from $\mathcal{A}$.

If $\varphi_a(q) = p \in U_a$, define $n(p)$ by

$$n(p) = \left.\frac{\partial_u \varphi_a \times \partial_v \varphi_a}{|\partial_u \varphi_a \times \partial_v \varphi_a|}\right|_q$$

Assume ψ_b is another parametrization from $\mathcal{A}$ such that $p \in U_b$. Then

$$\psi_b = \psi_a \circ \underset{\overset{\shortparallel}{\Theta}}{\theta_{ab}} \quad \Longrightarrow$$

$$\left(\partial_{y_1}\psi_b,\ \partial_{y_2}\psi_b\right) = \left(\partial_{x_1}\psi_a,\ \partial_{x_2}\psi_b\right) \cdot D\Theta$$

$$\Longrightarrow \quad \partial_{y_1}\psi_b \times \partial_{y_2}\psi_b = \underset{\overset{\vee}{0}}{\det(D\Theta)} \cdot \partial_{x_1}\psi_a \times \partial_{x_2}\psi_b$$

$\Longrightarrow$ $n(p)$ does not depend on the choice of parametrization near p

Since n is smooth in a nbhd of p, n is smooth everywhere. $\blacksquare$