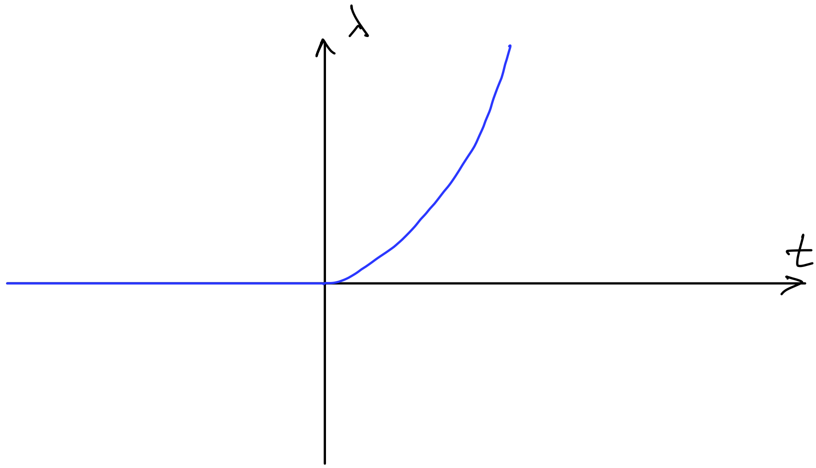
Partitions of unity

Recall that the function $\lambda : \mathbb{R} \to \mathbb{R}$

$$\lambda(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ e^{-1/t} & \text{if } t > 0 \end{cases}$$

is smooth.



For any fixed $r > 0$ we have

$$\lambda(t) + \lambda(r-t) > 0 \qquad \forall t \in \mathbb{R}$$

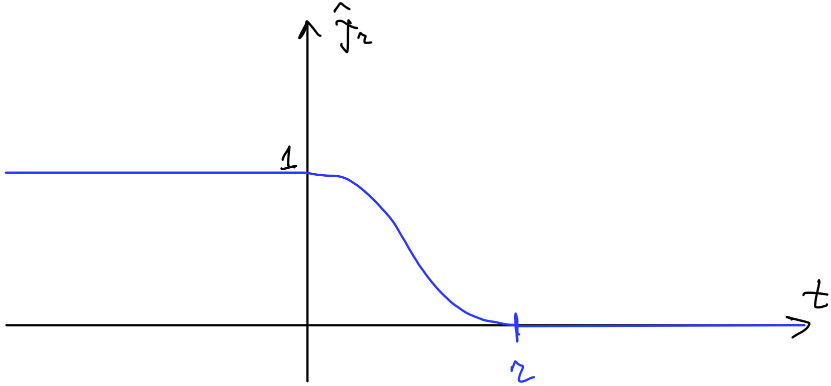
positive for $t > 0$

positive for $r - t > 0 \iff t < r$

Define

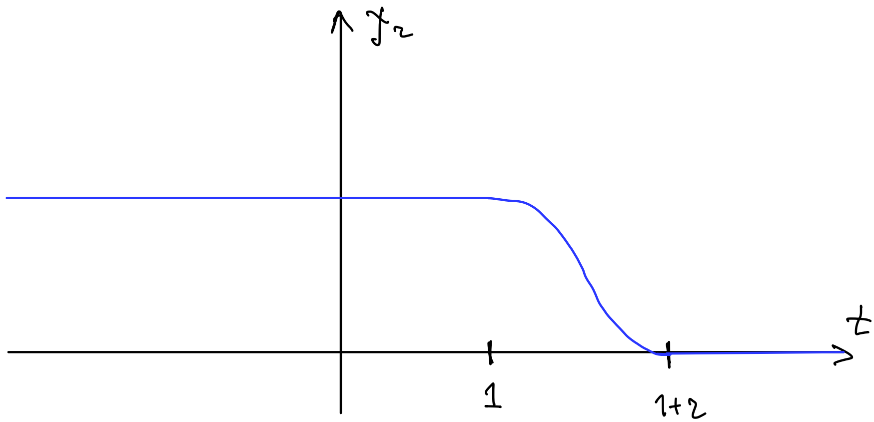
$$\hat{\chi}_r(t) := \frac{\lambda(r-t)}{\lambda(t) + \lambda(r-t)},$$

which is smooth everywhere on $\mathbb{R}$.



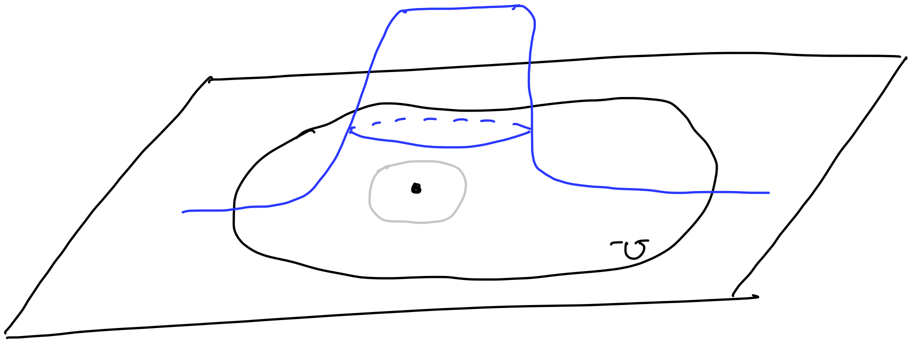
Denote also

$$\chi_r(t) := \hat{\chi}_r(t-1)$$



<u>Lemma</u> For any pt $p \in \mathbb{R}^n$ and any nbhd $U \ni p$ there exists a nbhd $V \subset U$ and $\rho \in C^\infty(\mathbb{R}^n)$ s.t. the following holds:

- $0 \le \rho(x) \le 1 \quad \forall x \in \mathbb{R}^n$
- $\rho|_{\bar{V}} \equiv 1$ and $\rho|_{\mathbb{R}^n \setminus \bar{U}} \equiv 0$.



Schematic graph of ρ

Proof For any $R>0$, consider

$$\rho(x) := \chi_1\left(\frac{|x-p|}{R}\right).$$

If $B_{2R}(p) \subset U$, then ρ vanishes

the ball of radius $2R$ centered at p

outside of $B_{2R}(p)$, so vanishes outside of U.
Also, $\rho(x) \equiv 1$ on $B_R(p)$ and $\rho \in C^\infty$. $\square$

Def For a continuous function f on a topological space X the support of f is

$$\operatorname{supp} f = \overline{\{x \in X \mid f(x) \neq 0\}}$$

In particular, $x \notin \operatorname{supp} f \implies f(x) = 0$

$$\not\Longleftarrow$$

Example

1) $\operatorname{supp} \lambda = [0, +\infty)$. Notice that $0 \in \operatorname{supp} \lambda$ although $\lambda(0) = 0$.

2) If ρ is as in the above lemma, then $\operatorname{supp} \rho \subset U$.

3) For $f(x) = |x|^2 - 1$, $f : \mathbb{R}^n \to \mathbb{R}$, $\operatorname{supp} f = \mathbb{R}^n$.

Def A (smooth) partion of unity on $\mathbb{R}^n$ is a family of smooth functions $\{\rho_\alpha \mid \alpha \in A\}$ s.t.

(i) $0 \le \rho_\alpha(x) \le 1 \quad \forall x \in \mathbb{R}^n \quad \forall \alpha \in A$

(ii) For any $x \in \mathbb{R}^n$ $\rho_\alpha(x) \ne 0$ for finitely many $\alpha \in A$ only.

(iii) $\sum_{\alpha \in A} \rho_\alpha(x) = 1 \quad \forall x \in \mathbb{R}^n$.

Rem More precisely, (ii) in the above definition should be replaced by the following condition:

$\forall x \in \mathbb{R}^n$ $\exists$ a nbhd $V \ni x$ s.t. the set

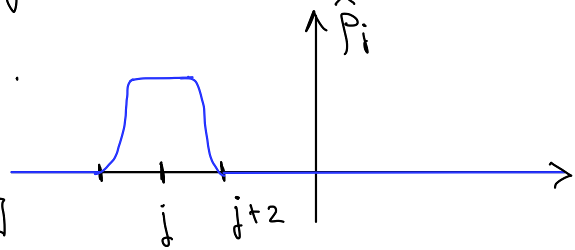
$$\{\alpha \in A \mid \operatorname{supp} \rho_\alpha \cap V = \emptyset\} \text{ is finite.}$$

However, we consider mostly finite partitions of unity so that this condition (and therefore, also (ii)) will be satisfied automatically.

<u>Example</u> (A partition of unity on $\mathbb{R}^1$)

Consider $\{\hat{\rho}_j(x) \mid j \in \mathbb{Z}\}$, where

$$\hat{\rho}_j(x) = \chi_1(|x-j|).$$



Notice that

$$\operatorname{supp} \hat{\rho}_j \subset [j-2, j+2]$$

Consider

$$\hat{\rho}(x) = \sum_{j \in \mathbb{Z}} \hat{\rho}_j(x)$$

well-defined, smooth and positive everywhere

Therefore

$$\{ \rho_j = \hat{\rho}_j / \hat{\rho} \mid j \in \mathbb{Z} \}$$

is a partition of unity on $\mathbb{R}^1$.

Just like for $\mathbb{R}^n$, the partition of unity is defined for surfaces.

<u>Theorem</u> (Existence of a partition of unity)
Let $\mathcal{U} = \{ U_\alpha \mid \alpha \in A \}$ be any open covering of a surface S. Then $\exists$ a partition of unity $\{ \rho_\beta \mid \beta \in B \}$ s.t. $\forall \beta$

$$\operatorname{supp} \rho_\beta \subset U_\alpha$$

for some $\alpha \in A$.

<u>Proof</u> The proof is given for compact surfaces only.

<u>Step 1</u>. Let S be any surface. For any $p \in S$ and any open $W \subset S$, $p \in W$, there exist $\rho \in C^\infty(S)$ s.t.

(i) $0 \leq \rho(q) \leq 1 \qquad \forall q \in S$

(ii) $\operatorname{supp} \rho \subset W$.

(iii) $\exists X \subset W$ open s.t. $\rho|_X \equiv 1$.

Let (U, φ) be a chart on S s.t. $\varphi(p) = 0 \in V \subset \mathbb{R}^2$ and $U \subset W$.

Pick a function $\hat{\rho} \in C^\infty(\mathbb{R}^2)$ s.t.

$$0 \le \hat{\rho} \le 1, \quad \hat{\rho}|_{B_r(0)} \equiv 1, \quad \hat{\rho}|_{\mathbb{R}^2 \setminus B_{2r}(0)} \equiv 0$$

for some $r > 0$ s.t. $B_{2r}(0) \subset V$.

Define

$$\rho(p) := \begin{cases} \hat{\rho} \circ \varphi(p), & p \in U. \\ 0, & p \notin U. \end{cases}$$

Then ρ is smooth everywhere and with $X := \varphi^{-1}(B_r(0))$ satisfies (i) - (iii).

<u>Alternatively</u>: One can first define a suitable function $\tilde{\rho}$ on a nbhd of p in $\mathbb{R}^3$ and define ρ as the restriction of $\tilde{\rho}$ to S.

<u>Rem</u> The function constructed in Step 1 is called a bump function.

<u>Step 2</u> We prove this thm assuming S is compact

Pick any U_α and any $p \in U_\alpha$. Then $\exists$ a chart $(U_{p,\alpha}, \varphi_{p,\alpha})$ s.t. $U_{p,\alpha} \subset U_\alpha$.

By Step 1, $\exists$ $X_{p,\alpha} \subset U_{p,\alpha}$ and a function $\hat{\rho}_{p,\alpha}$ satisfying (i) - (iii).

Consider the family $\{ X_{p,\alpha} \mid p \in S, \alpha \in A \}$, which is an open covering of S.
By the compactness of S, $\exists$ a finite subcovering

$$\begin{matrix} X_{p_1,\alpha_1}, & \cdots & , & X_{p_n,\alpha_n} \\ \| & & & \| \\ X_1 & & & X_n \end{matrix}$$

Denote $\hat{\rho}_j := \hat{\rho}_{p_j,\alpha_j}$ so that $\hat{\rho}_j |_{X_j} \equiv 1$

and consider

$$\hat{\rho}(p) := \sum_{j=1}^{n} \hat{\rho}_j(p) > 0 \quad \forall p \in S.$$

Then $\rho_j := \hat{\rho}_j / \hat{\rho}$ is a partition of unity on S. Moreover,

$$\operatorname{supp} \rho_j = \operatorname{supp} \hat{\rho}_j \subset U_j \subset U_{\alpha_j} \qquad \boxed{///}$$

<u>Rem</u> A partition of unity as in the above theorem is called <u>subordinate</u> to $\mathcal{U}$.

<u>Example</u> $S = S^2$, $\mathcal{U} = \{ S^2 \setminus \{N\}, S^2 \setminus \{S\} \}$

Let ρ be a bump function on $\mathbb{R}^2$ s.t. $\rho|_{B_1(0)} \equiv 1$ and $\operatorname{supp} \rho \subset B_2(0)$.

Define $\rho_N := \rho \circ \varphi_N$

$$\rho_S := 1 - \rho_N$$

Then $\{ \rho_N, \rho_S \}$ is a partition of unity on S^2.

<u>Integration on surfaces</u>

<u>Aim</u>: Define a map $\int : C^\infty(S) \to \mathbb{R}$ with "the usual" properties of the integral, e.g.

$$(*) \quad \int_S (\lambda f + \mu g) = \lambda \int_S f + \mu \int_S g \qquad \begin{array}{l} \lambda, \mu \in \mathbb{R} \\ f, g \in C^\infty(S) \end{array}$$

We assume in addition that S is compact.

Chose an atlas $\mathcal{A} = \{ (U_\alpha, \varphi_\alpha) \mid \alpha \in A \}$ on S. Let $\{ \rho_j \mid j = 1, \dots, J \}$ be a partition of unity on S s.t. $\operatorname{supp} \rho_j \subset U_{\alpha_j} =: U_j$

For any $f \in C^\infty(S)$ we have

$$f = f \cdot 1 = \sum_{j=1}^{I} \underbrace{f \cdot \rho_j}_{=: f_j} = \sum_j f_j$$

and $\operatorname{supp} f_j \subset \operatorname{supp} \rho_j \subset U_j$.

Hence, by (8.*) it suffices to define $\int_S f_j$ that is we want to define

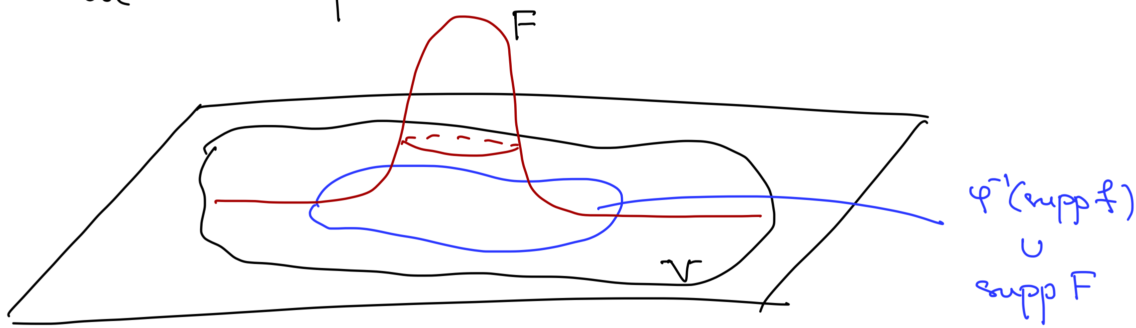
$$\int_S f \quad \text{provided} \quad \operatorname{supp} f \subset U,$$

where (U, φ) is a chart.

Viewing φ as an identification between U and $V \subset \mathbb{R}^2$, we can identify f with its coordinate representation

$$F := f \circ \varphi^{-1} = f \circ \psi : V \longrightarrow \mathbb{R}.$$

Then F vanishes outside of $\varphi^{-1}(\operatorname{supp} F)$, which is cmpt.



It is tempting to define

$$\int_S f := \int_{\mathbb{R}^2} F(u,v)\, du\, dv. \qquad (*)$$

It may happen, however, that there is another chart $(\hat{U}, \hat{\varphi})$ on S s.t.

$$\operatorname{supp} f \subset \hat{U}$$

To show that $\int_S f$ is well-defined, we must show the equality

$$\int_{\mathbb{R}^2} F(u,v)\, du\, dv \overset{?}{=} \int_{\mathbb{R}^2} \hat{F}(x,y)\, dx\, dy, \qquad (**)$$

where $\hat{F} = f \circ \hat{\varphi}^{-1}$ is the coord. rep. of f with respect to $\hat{\varphi}$.

Let $\Theta = \varphi \circ \hat{\varphi}^{-1} \iff (u,v) = \Theta(x,y)$ denote the change of coordinates map. Then

$$\hat{F} = f \circ \hat{\varphi}^{-1} = f \circ \varphi^{-1} \circ \varphi \circ \hat{\varphi}^{-1} = F \circ \Theta,$$

so that $(**)$ is equivalent to

$$\int_{\mathbb{R}^2} F(u,v)\, du\, dv \overset{?}{=} \int_{\mathbb{R}^2} F \circ \Theta(x,y)\, dx\, dy$$

The last equality is false in general, since by a thm from analysis

$$\int_{\mathbb{R}^2} F(u,v)\,du\,dv = \int_{\mathbb{R}^2} F\circ\Theta(x,y)\,|\det D\Theta|\,dx\,dy$$

Thus, our naïve approach to define $\int_S f$ by (10.$*$) is false in general.

To solve this problem, recall the following fact. Suppose $V \subset \mathbb{R}^3$ be a bounded open set such that $S := \partial V$ is a smooth surface. Then

$$\int_V \operatorname{div} v = \int_S \langle v, n\rangle\, dS$$

where n is the unit normal field pointing outwards. If $\varphi = \varphi(u,v)$ is a parametrization of S, the right hand side is defined by

$$\int \langle v, n\rangle\, |\partial_u\varphi \times \partial_v\varphi|\, du\,dv$$

Following this hint, for $f \in C^\infty(S)$ with supp $f \subset U$, where U is a coord. chart, we define

$$\int_S f := \int_{\mathbb{R}^2} F(u,v)\, |\partial_u \Psi \times \partial_v \Psi|\, du\, dv \quad (*)$$

Then, if $(\hat{U}, \hat{\varphi})$ is another chart just like above, we have

$$\hat{F} = F \circ \Theta, \qquad \Theta = \varphi \circ \hat{\varphi}^{-1} = \Psi^{-1} \circ \hat{\Psi}$$

$$\hat{\Psi} = \Psi \circ \Theta \Rightarrow$$

$$(\partial_x \hat{\Psi}, \partial_y \hat{\Psi}) = (\partial_u \Psi, \partial_v \Psi) \cdot D\Theta$$

$$\Rightarrow |\partial_x \hat{\Psi} \times \partial_y \hat{\Psi}| = |\partial_u \Psi \times \partial_v \Psi| \cdot |\det D\Theta|$$

Hence, we have

$$\int_{\mathbb{R}^2} \hat{F}(x,y)\, |\partial_x \hat{\Psi} \times \partial_y \hat{\Psi}|\, dx\, dy =$$

$$= \int_{\mathbb{R}^2} F \circ \Theta(x,y)\, |\partial_u \Psi \times \partial_v \Psi|\, |\det D\Theta|\, dx\, dy$$

$$= \int_{\mathbb{R}^2} F(u,v)\, |\partial_u \Psi \times \partial_v \Psi|\, du\, dv.$$

That is (12.*) does not depend on the choice of the parametrization of S.

Thus, for any $f \in C^\infty(S)$ we may set

$$\int_S f := \sum_j \int_S f_j =$$

$$= \sum_j \int_{\mathbb{R}^2} F_j(u,v)\, |\partial_u \varphi \times \partial_v \varphi|\, du\,dv$$

<u>Prop</u> $\int_S f$ is well-defined, that is $\int_S f$ does not depend on the choice of an atlas.

<u>Proof</u> Let $\hat{\mathcal{U}} = \{(\hat{U}_\beta, \hat{\varphi}_\beta) \mid \beta \in B\}$ be another atlas on S. Choose a partition of unity $\{\mu_k \mid k = 1, \dots, K\}$ subordinate to $\hat{\mathcal{U}}$. We need to show that

$$\sum_j \int_S (\rho_j f) \overset{?}{=} \sum_k \int_S (\mu_k f)$$

Notice that $\{\lambda_{jk} := \rho_j \lambda_k \mid j = 1, \dots, J,\ k = 1 \dots K\}$ is also a partition of unity and

$$\operatorname{supp} \lambda_{jk} \subset U_j \cap \hat{U}_k .$$

With this understood, consider

$$\sum_{j=1}^{J} \sum_{k=1}^{K} \int_S \underbrace{\lambda_{jk}}_{=\rho_j \mu_k} f = \sum_{j=1}^{J} \int_S \Big(\rho_j \sum_{k=1}^{K} \mu_k f \Big) = \sum_{j=1}^{J} \int_S \rho_j f$$

$$\Big\|$$

$$\sum_{k=1}^{K} \sum_{j=1}^{J} \int_S (\lambda_{jk} f) = \sum_k \int_S \Big(\mu_k \sum_{j=1}^{J} \rho_j f \Big)$$

$$= \sum_k \int_S \mu_k f \qquad \square$$

It follows immediately from the definition that $\int_S$ has the usual properties known from the analysis course, for example:

- $\int_S (\lambda f + \mu g) = \lambda \int_S f + \mu \int_S g$;
- $f \geq 0 \implies \int_S f \geq 0$;
- $\int_S f = 0$ and $f \geq 0 \implies f \equiv 0$

and so on.

Ex Let $f: S^2 \to \mathbb{R}$ be any (smooth) function. Let $\mathcal{U} = \{ S^2 \setminus \{N\}, S^2 \setminus \{S\} \}$ be just as in the example on P. 8. Choose $\varepsilon > 0$ and set

$$\rho_N^\varepsilon(p) := \rho(\varepsilon\, \varphi_N(p)),$$

$$\rho_S^\varepsilon := 1 - \rho_N^\varepsilon,$$

where ρ is just as in the example on P. 8

Notice the following:

$$\rho|_{B_1(0)} \equiv 1 \implies \rho_N^\varepsilon \big|_{\varphi_N^{-1}(B_{\varepsilon^{-1}}(0))} \equiv 1$$

$$\rho\,\big|_{\mathbb{R}^2 \setminus B_2(0)} \implies \rho_N^\varepsilon \big|_{S^2 \setminus \varphi_N^{-1}(B_{2\varepsilon^{-1}}(0))} \equiv 0$$

If $F_N = f \circ \psi_N$ and $F_S := f \circ \psi_S$ are coordinate representations of f, then by the definition of the integral we have

$$\int_S f = \int_{\mathbb{R}^2} \left(\rho_N^\varepsilon \circ \psi_N(u,v)\right) F_N(u,v)\, |\partial_u \psi_N \times \partial_v \psi_N|\, du\,dv$$

$$+ \int_{\mathbb{R}^2} \left(\rho_S^\varepsilon \circ \psi_S(u,v)\right) \cdot F_S(u,v)\, |\partial_u \psi_S \times \partial_v \psi_S|\, du\,dv$$

$$= \int_{\mathbb{R}^2} \rho(\varepsilon u, \varepsilon v)\, F_N(u,v)\, |\partial_u \Psi_N \times \partial_v \Psi_N|\, du\, dv$$

$$+ \int_{\mathbb{R}^2} \rho_S^\varepsilon \circ \Psi_S(u,v)\, F_S(u,v)\, |\partial_u \Psi_S \times \partial_v \Psi_S|\, du\, dv$$

The last term converges to 0 as $\varepsilon \to 0$, since

- the measure of the support of $\rho_S^\varepsilon \circ \Psi_S$ converges to zero;
- the integrand is uniformly bounded with respect to ε.

For the first term, we have

$$\int_{\mathbb{R}^2} \rho(\varepsilon u, \varepsilon v)\, F_N(u,v)\, |\partial_u \Psi_N \times \partial_v \Psi_N|\, du\, dv$$

$$= \int_{B_{\varepsilon^{-1}}(0)} F_N(u,v)\, |\partial_u \Psi_N \times \partial_v \Psi_N|\, du\, dv$$

$$+ \int_{B_{2\varepsilon^{-1}}(0) \setminus B_{\varepsilon^{-1}}(0)} \rho(\varepsilon u, \varepsilon v)\, F_N(u,v)\, |\partial_u \Psi_N \times \partial_v \Psi_N|\, du\, dv$$

The last summand of this expression converges to zero, since

- $|\rho(\varepsilon u, \varepsilon v) \; F_N(u,v)| \leq \sup_{S^2} |f|$
- $\int_{B_{2\varepsilon^{-1}}(0) \setminus B_{\varepsilon^{-1}}(0)} |\partial_u \varphi \times \partial_v \varphi| \, du\, dv \leq \text{Area}(S^2) = 4\pi.$

Summing up, we obtain

$$\int_{S^2} f = \int_{\mathbb{R}^2} F_N(u,v) \, |\partial_u \varphi_N \times \partial_v \varphi_N| \, du\, dv \quad (*)$$

just as it is well-known from the analysis course.

Of course, a similar argument yields also

$$\int_{S^2} f = \int_{\mathbb{R}^2} F_S(u,v) \, |\partial_u \varphi_S \times \partial_v \varphi_S| \, du\, dv. \quad (**)$$

The reader should check directly that the right hand sides of $(*)$ and $(**)$ are equal indeed.

Thm Let $h: S_1 \to S_2$ be a diffeomorphism, where S_1 and S_2 are compact surfaces. Then for any $f \in C^\infty(S)$ we have

$$\int_{S_2} f = \int_{S_1} (f \circ h) \cdot |\det dh| \qquad (*)$$

To explain the right hand side, let V and W be Euclidean vector spaces such that $\dim V = \dim W = n$. Choose an orthonormal basis $e = (e_1, \dots, e_n)$ of V and an orthonormal basis $g = (g_1, \dots, g_n)$ of W. A linear map $\varphi : V \to W$ can be represented by a matrix $A_\varphi \in M_n(\mathbb{R})$, where

$$A_\varphi = (a_{ij}) \qquad \varphi(e_i) = \sum_{j=1}^{n} a_{ij} g_j$$

$$\iff (\varphi(e_1), \dots, \varphi(e_n)) = (g_1, \dots, g_n) \cdot A$$

$$\iff \varphi(e) = g \cdot A$$

If e' is another basis of V, then $\exists$ an orthogonal $n \times n$ matrix B s.t.

$$e' = e \cdot B \iff e'_i = \sum_{j=1}^{n} b_{ij} e_j$$

Similarly, if g' is another basis of W, then there exists an orthogonal $n \times n$ matrix $C = (c_{ij})$ s.t.

$$g' = gC \iff g'_i = \sum_{j=1}^{n} c_{ij} g_j$$

Let A'_φ be the matrix of φ with respect to e' and g'. Then

$$\varphi(e') = g' A'_\varphi = g C A'_\varphi$$

$$\varphi(e') = \varphi(e \cdot B) \underset{\substack{\text{linearity} \\ \text{of } \varphi}}{=} \varphi(e) \cdot B = g \cdot A_\varphi B$$

$$\Rightarrow \quad C A'_\varphi = A_\varphi B \quad \Rightarrow \quad \boxed{A'_\varphi = C^{-1} A_\varphi B}$$

Therefore

$$\det A'_\varphi = \det(C^{-1}) \det A_\varphi \det B$$

± 1 since B and C are orthogonal

$$= \pm \det A_\varphi$$

$$\Rightarrow \quad |\det A'_\varphi| = |\det A_\varphi|$$

That is for any linear map $\varphi : V \to W$ between Euclidean spaces $|\det \varphi| := |\det A_\varphi|$ is well-defined.

Since for any $p \in S_1$ both $T_p S_1$ and $T_{h(p)} S_2$ are Euclidean, $|\det dh|$ is a well-defined function on S_1.

<u>Proof of the theorem</u>

Let $\mathcal{U}_2 = \{ (V_\alpha, \varphi_\alpha) \mid \alpha \in A \}$ be an atlas on S_2. Pick a partition of unity $\{ \rho_j \mid j = 1, \dots, n \}$ on S_2 subordinate to $\mathcal{U}_2$.

Then
$$\mathcal{U}_1 = \{ (h^{-1}(V_\alpha), \underbrace{\varphi_\alpha \circ h}_{\xi_\alpha}) \mid \alpha \in A \}$$

is an atlas on S_1 and $\{ \rho_j \circ h \mid j = 1, \dots, n \}$ is a partition of unity subordinate to $\mathcal{U}_1$.

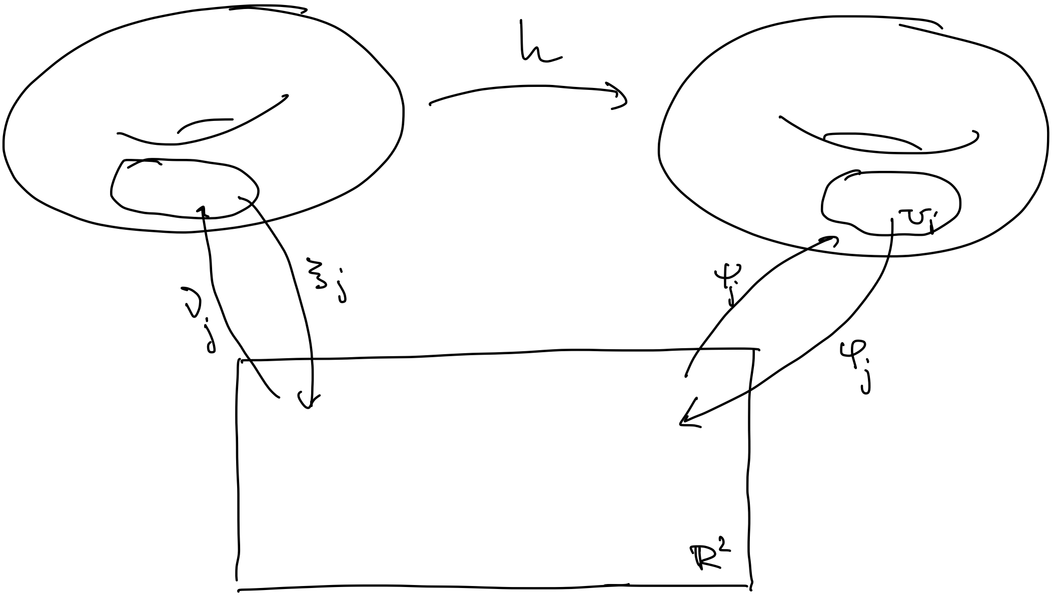
If $\operatorname{supp} \rho_j \subset V_{\alpha_j} =: V_j$, denote $\psi_j = \varphi_j^{-1}$

$\xi_j = \varphi_{\alpha_j} \circ h$ and $\vartheta_j = \xi_j^{-1} = h^{-1} \circ \psi_j$

$$\iff \psi_j = h \circ \vartheta_j \implies \begin{aligned} \partial_u \psi_j &= dh(\partial_u \vartheta_j) \\ \partial_v \psi_j &= dh(\partial_v \vartheta_j) \end{aligned}$$

$$\implies |\partial_u \psi_j \times \partial_v \psi_j| = |\det dh| \; |\partial_u \vartheta_j \times \partial_v \vartheta_j|$$

follows from: $A : \mathbb{R}^3 \to \mathbb{R}^3$ linear $\implies$
$(Av) \times (Aw) = (\det A) \cdot v \times w$



$$\int_{S_1} (\rho_j \circ h) \cdot (f \circ h) \cdot |\det dh| =$$

$$= \int_{\mathbb{R}^2} (\underbrace{\rho_j \circ h \circ \xi_j^{-1}}_{\psi_j}) \cdot (\underbrace{f \circ h \circ \xi_j^{-1}}_{\psi_j}) \cdot \underbrace{|\det dh| \cdot |\partial_u \vartheta_j \times \partial_v \vartheta_j|}_{|\partial_u \psi_j \times \partial_v \psi_j|}$$

$$= \int_{\mathbb{R}^2} (\rho_j \circ \psi_j) \cdot (f \circ \psi_j) \, |\partial_u \psi_j \times \partial_v \psi_j|$$

$$= \int_{S_2} \rho_j \cdot f$$

Summing up by j, we obtain (15.*) $\boxtimes$

<u>Rem</u> Notice that (15.*) is nothing else but a fancy restatement of the theorem about the change of coordinates for the integration, which is well-known from the analysis course.