Quadratic forms on surfaces

Let S be a surface.

Def A Riemannian metric on S is a family of scalar products $\langle \cdot, \cdot \rangle_p$ on each tangent space T_pS, $p \in S$, such that $\langle \cdot, \cdot \rangle_p$ depends smoothly on p.

To explain, let $\varphi : V \to U$ be a parametrization. If $q \in V$ and $p = \varphi(q)$, then T_pS has a basis $(\partial_u \varphi, \partial_v \varphi)$. Hence, the scalar product $\langle \cdot, \cdot \rangle_p$ is represented by its Gram matrix

$$M = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \qquad \begin{aligned} E &= \langle \partial_u \varphi, \partial_u \varphi \rangle_p \\ F &= \langle \partial_u \varphi, \partial_v \varphi \rangle_p \\ G &= \langle \partial_v \varphi, \partial_v \varphi \rangle_p \end{aligned}$$

We say, that $\langle \cdot, \cdot \rangle_p$ depends smoothly on p, if all 3 functions E, F, G are smooth (on V, where they are defined).

Ex For any $p \in S$ we have $T_pS \subset \mathbb{R}^3$. Since $\mathbb{R}^3$ is equipped with the standard scalar product

$$\langle x, y \rangle_{st} := x_1 y_1 + x_2 y_2 + x_3 y_3$$

we can restrict $\langle\cdot,\cdot\rangle_{st}$ to T_pS to obtain a scalar product on T_pS. This is a Riemannian metric on S, since

$$E(u,v) = \langle \partial_u\psi, \partial_u\psi\rangle_S = \langle \partial_u\psi, \partial_u\psi\rangle_{st}$$

is a smooth function of (u,v) (and similarly for F and G).

This particular Riemannian metric on S is called <u>the first fundamental form</u> of S in the classical theory of surfaces.

<u>Exercise</u> Let $\langle\cdot,\cdot\rangle$ be the first fundamental form of S and $f: S \to S$ be a diffeomorphism. For $v, w \in T_pS$ define a new scalar product

$$\langle v, w\rangle_f := \langle \underset{\substack{\cap \\ T_{f(p)}S}}{d_pf(v)}, \underset{\substack{\cap \\ T_{f(p)}S}}{d_pf(w)}\rangle_{f(p)}$$

Show that $\langle\cdot,\cdot\rangle_f$ is a Riemannian metric on S.

For the sake of simplicity of exposition, assume S is oriented and let n be the unit normal field. We can view n as a smooth map

$$n : S \longrightarrow S^2,$$

which is called <u>the Gauss map</u>. Then $\forall p \in S$ we have

$$d_p n : T_p S \longrightarrow T_{n(p)} S^2 = n(p)^{\perp} = T_p S.$$

This is called <u>the shape operator</u>.

As a linear map in a 2-dimensional vector space, the shape operator has two invariants:

$$\boxed{K(p) := \det(d_p n)} \quad \text{and} \quad H(p) := -\tfrac{1}{2} \operatorname{tr}(d_p n)$$

<u>Def</u> $K(p)$ is called the Gauss curvature and $H(p)$ is called the mean curvature of S at p.

K, H are smooth functions on S.

<u>Ex 1</u> $S = \mathbb{R}^2 \equiv \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$.

Gauss map $n(p) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ constant

Shape operator $d_p n \equiv 0$

$\Rightarrow \quad K \equiv 0.$

Ex 2 $\quad S^2_r := \{ x \in \mathbb{R}^3 \mid |x|^2 = r^2 \}$

Gauss map $\quad n(p) = \frac{1}{r} p$

The shape operator: $\quad d_p n(v) = \frac{1}{r} v \Rightarrow d_p n = \frac{1}{r} id$

$\Rightarrow \quad K(p) = \frac{1}{r^2}$ is constant on S^2

If $r \to \infty$, $K(p) \to 0$ and the sphere looks more and more flat in a nbhd of each point (that is why our Earth is "flat").

Thus, we can view the Gauss curvature as a measure of flatness of S.

Lemma The shape operator is symmetric, that is

$$\langle d_p n(v), w \rangle = \langle v, d_p n(w) \rangle$$

$\forall p \in S$ and $\forall v, w \in T_pS$.

Proof Let $\varphi : V \to S$ be a parametrization s.t. $\varphi(0) = p$. Then $(\partial_u \varphi, \partial_v \varphi)|_{(u,v)=0}$ is a basis of T_pS. Hence, it suffices to show the equaility

$$\langle d_p n(\partial_u \varphi), \partial_v \varphi \rangle = \langle \partial_u \varphi, d_p n(\partial_v \varphi) \rangle, \quad (*)$$

where the derivatives are evaluated at the origin.

To this end, notice that by the definition of n we have

$$\langle n(\varphi(u,v)), \partial_u \varphi(u,v) \rangle = 0 \qquad \forall (u,v) \in V$$

Differentiating this equality with respect to v and setting $(u,v) = 0$, we obtain

$$\langle d_p n(\partial_u \varphi), \partial_v \varphi \rangle + \langle n(p), \partial_{uv} \varphi \rangle = 0$$

Similarly, we obtain

$$\langle \partial_u \varphi, d_p n(\partial_v \varphi) \rangle + \langle \partial_{uv} \varphi, n(p) \rangle = 0.$$

Subtracting these two equalities, we arrive at (4.*). $\square$

<u>Def</u> The bilinear symmetric map

$$\mathrm{II} : T_p S \times T_p S \longrightarrow \mathbb{R}$$

$$(v, w) \longmapsto \langle v, d_p n(w) \rangle_p$$

is called <u>the second fundamental form</u> of S (at the point p).

Notice that II is smooth, that is for any parametrization φ

$$\mathrm{II}(\partial_u \varphi(u,v), \partial_u \varphi(u,v)), \quad \mathrm{II}(\partial_u \varphi, \partial_v \varphi),$$

$$\mathrm{II}(\partial_v \varphi, \partial_v \varphi)$$

are smooth functions of (u,v).

<u>Rem</u> One can recover the shape operator from the second fundamental form, that is these two objects contain the same ammount of information.

The geometric meaning of the sign of the Gauss curvature.

Let $p \in S$ be a critical pt of $f \in C^\infty(S)$. Given $v \in T_pS$, pick $\gamma : (-\varepsilon, \varepsilon) \to S$ s.t. $\gamma(0) = p$ and $\dot\gamma(p) = v$.

<u>Def</u> The map

$$\mathrm{Hess}_p f : T_pS \to \mathbb{R}, \quad \mathrm{Hess}_p f\,(v) = \frac{d}{dt}\Big|_{t=0} \big(f \circ \gamma(t)\big)$$

is called the Hessian of f at p.

<u>Prop</u>

(i) $\mathrm{Hess}_p f$ is a well-defined quadratic map;

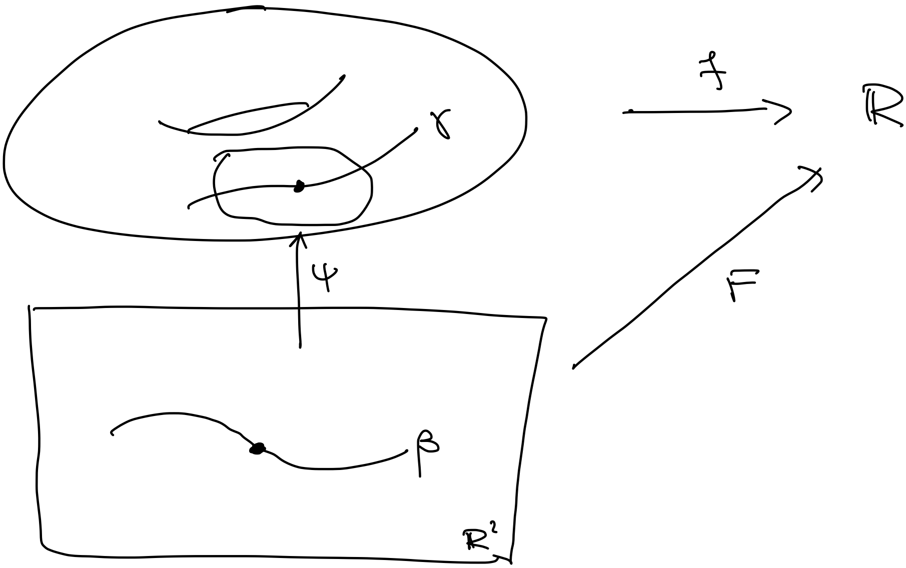
(ii) If p is a pt of loc. minimum, then $\mathrm{Hess}_p(f)\,(v) \geq 0 \quad \forall v \in T_pS$. If p is a pt of loc. maximum, then $\mathrm{Hess}_p f\,(v) \leq 0$.

(iii) If $\mathrm{Hess}_p f\,(v) > 0 \quad \forall v \neq 0$, then p is a pt of loc minimum. If $\mathrm{Hess}_p f\,(v) < 0 \quad \forall v \neq 0$, then p is a pt of loc. maximum.

Proof

Choose a parametrization ψ s.t. $\psi(0)=p$ and denote

$$F := f\circ\psi \qquad \beta := \varphi\circ\gamma = \psi^{-1}\circ\gamma.$$



Then if $\beta(t) = (\beta_1(t), \beta_2(t))$, we have

$$f\circ\gamma(t) = F\circ\beta(t) = F(\beta_1(t), \beta_2(t))$$

$$\Rightarrow \frac{d}{dt} f\circ\gamma(t) = \partial_u F(\beta(t))\,\beta_1'(t) + \partial_v F(\beta(t))\,\beta_2'(t)$$

Notice that $\beta(0)=0$ and $\partial_u F(0) = 0 = \partial_v F(0)$.
Furthermore we have

$$\left.\frac{d^2}{dt^2}\right|_{t=0} f\circ\gamma(t) = \partial^2_{uu}F(0)\,\beta_1'(0)^2 + 2\partial^2_{uv}F(0)\,\beta_1'(0)\,\beta_2'(0) + \partial^2_{vv}F(0)\,\beta_2'(0)^2. \qquad (*)$$

Recalling that $\beta'(0) = d_p\varphi(v)$, we see that the right-hand-side of (7.*) depends only on $\beta'(0)$ and not on the choice of γ.

Moreover, (7.*) also shows that $\mathrm{Hess}_p f(v)$ is a quadratic form of v.

In fact we have shown that $\mathrm{Hess}_p f$ corresponds to the Hessian of the loc. representation F of f in the following sense: The diagram

$$\begin{tikzcd} T_pS \arrow[r, "\mathrm{Hess}_p f"] \arrow[d, "d_p\varphi"'] & \mathbb{R} \\ \mathbb{R}^2 \arrow[ur, "\mathrm{Hess}_{\varphi(p)} F"'] & \end{tikzcd}$$

commutes. That is we can identify $\mathrm{Hess}_p f$ with $\mathrm{Hess}_{\varphi(p)} F$ by means of the isomorphism $d_p\varphi : T_pS \to \mathbb{R}^2$. This immediately implies (ii) and (iii). $\square$

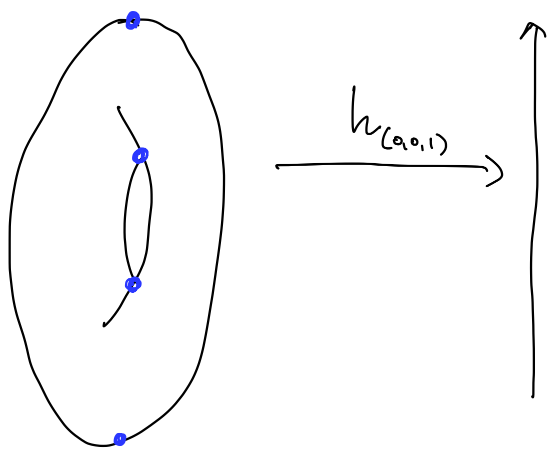
Let $a \in \mathbb{R}^3$ be any fixed vector, $a \neq 0$.
Let $h_a : S \to \mathbb{R}$ be the restriction of

$$\mathbb{R}^3 \to \mathbb{R}, \quad x \longmapsto \langle x, a \rangle.$$

Then h_a is called the height function on S in the direction of a.

Notice that p is a critical pt of h_a if and only if $T_pS \perp a$.

Ex For $a = (0,0,1)$ we have the standard height function



Prop Let n be an orientation of S. Then for any $p \in S$ we have

$$\mathrm{II}_p = -\,\mathrm{Hess}_p\left(h_{n(p)}\right)$$

<u>Proof</u> Observe first that

$T_pS \perp n(p)$ that is p is a critical pt of $h_{n(p)}$.

Given $v \in T_pS$ choose a curve $\gamma : (-\varepsilon, \varepsilon) \to S$ s.t. $\gamma(0) = p$ and $\dot\gamma(0) = v$. Then

$$\mathrm{Hess}_p(h_{n(p)}) = \frac{d^2}{dt^2}\Big|_{t=0} \langle \gamma(t), n(p) \rangle$$

$$= \langle \ddot\gamma(0), n(p) \rangle$$

However, $\gamma(t) \in S \Rightarrow \dot\gamma(t) \in T_{\gamma(t)}S \quad \forall t$

$$\Rightarrow \langle \dot\gamma(t), n(\gamma(t)) \rangle = 0 \quad \forall t$$

$$\frac{d}{dt}\Big|_{t=0} \qquad \langle \ddot\gamma(0), n(p) \rangle + \underbrace{\langle \dot\gamma(0), d_pn(\dot\gamma(0)) \rangle}_{\mathrm{II}_p(v)} = 0$$

This yields $\mathrm{II}_p(v) = -\langle \ddot\gamma(0), n(p) \rangle$

$$= -\mathrm{Hess}_p(h_{n(p)}) \qquad \boxtimes$$

Fix $p \in S$. Without loss of generality assume that

$$p = 0 \in \mathbb{R}^3 \quad \text{and} \quad n(0) = (0,0,1).$$

This can be always achieved by applying

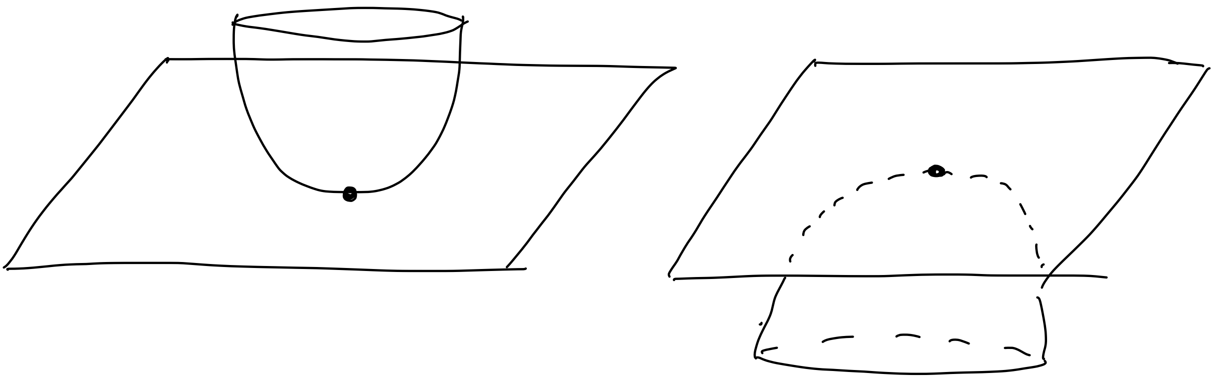
a translation and a rotation in $\mathbb{R}^3$.

Since the shape operator $d_o n : \underset{\mathbb{R}^2}{\underset{\parallel}{T_o S}} \longrightarrow \underset{\mathbb{R}^2}{\underset{\parallel}{T_o S}}$ is symmetric, $d_o n$ has two real eigenvalues, say k_1 and k_2.

Consider the following cases:

A) $K(p) > 0 \implies k_1 \cdot k_2 > 0 \implies$

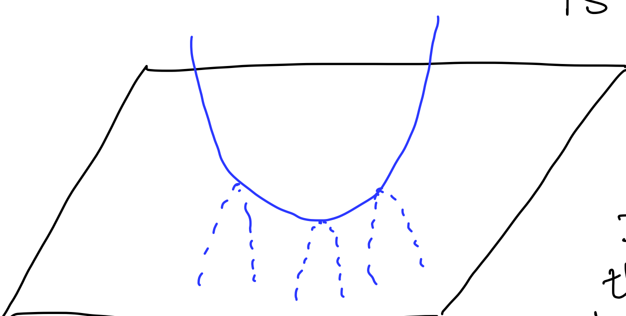
$\mathrm{Hess}_0(h_{n(0)})$ is either positive-definite or negative definite



B) $K(p) < 0 \implies z\big|_S$ attains both positive and negative values

$\Downarrow$

In any nbhd of p there are pts in S above and below $T_p S$.



<u>Rem</u> If $K(p) = 0$, in general one cannot say anything about the position of S relative to T_pS.

Surfaces of positive curvature and the Gauss-Bonnet theorem

Let S be a smooth connected surface.

<u>Thm</u> (Jordan separation thm)
If S is closed as a subset of $\mathbb{R}^3$, then $\mathbb{R}^3 \setminus S$ has exactly two connected components, whose common boundary is S. ▨

<u>Rem</u> The Jordan separation theorem is a well-known result from topology. Its proof requires certain results from topology, which are typically not proved in a standard course in topology. Hence, we take the Jordan separation thm as granted. An interested reader may find a proof in the book of Montiel-Ros (Thm. 4.16).

If S is compact, then one and only one component of $\mathbb{R}^3 \setminus S$ is bounded. This bounded open domain is called <u>the inner domain</u> of S. The unbounded domain is called <u>the outer dom. of S</u>.

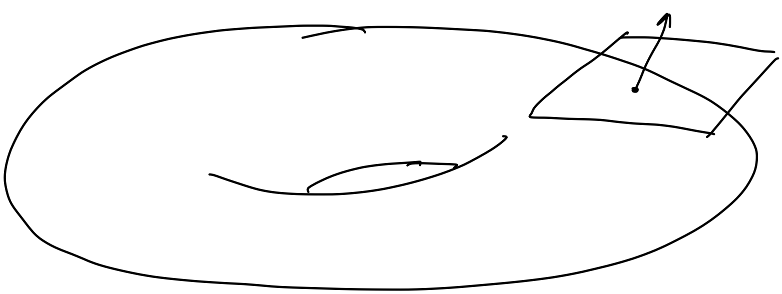
<u>Corollary</u> Any compact surface in $\mathbb{R}^3$ is orientable.

<u>Proof</u> Let $S \subset \mathbb{R}^3$ be a compact surface. Without loss of generality we can assume that S is connected (otherwise, pick a connected component of S).

Pick a pt $p \in S$. A unit vector n, which is normal at p, is said to be <u>pointing outwards</u>, if $\exists\, \varepsilon > 0$ s.t.

$$p + tn \in \Omega_{out} \quad \forall\, t \in (0, \varepsilon).$$

outer domain of S.



Pick a nbhd W of p in $\mathbb{R}^3$ and a smooth function $\varphi : W \to \mathbb{R}$ s.t.

$$S \cap W = \varphi^{-1}(0) \quad \text{and} \quad \nabla\varphi(x) \neq 0 \quad \forall x \in W.$$

__Exercise__ Show that $\varphi|_{\Omega_{in} \cap W} < 0$ and $\varphi|_{\Omega_{out} \cap W} > 0$ (or the other way around).

In other words,

$$\Omega_{in} \cap W = \{\varphi < 0\} \quad \text{and} \quad \Omega_{out} \cap W = \{\varphi > 0\},$$

which we assume for the sake of definitness.

Since

$$\varphi(p + t\nabla\varphi(p)) = \underbrace{\varphi(p)}_{=0} + \underbrace{|\nabla\varphi(p)|^2}_{>0} \cdot t + o(t)^2 > 0$$

provided $t > 0$ is sufficiently small, we obtain that

$$\frac{\nabla\varphi(p)}{|\nabla\varphi(p)|}$$

is pointing outwards for any $p \in S \cap W$.
A similar argument shows that $-\dfrac{\nabla\varphi(p)}{|\nabla\varphi(p)|}$ is pointing inwards.

Let $\hat{W}$ be any other open subset of $\mathbb{R}^3$ and $\hat{\varphi} \in C^\infty(\hat{W})$ s.t.

$$S \cap \hat{W} = \hat{\varphi}^{-1}(0), \quad \nabla\hat{\varphi}(x) \neq 0 \quad \forall x \in \hat{W},$$

$$\Omega_{in} \cap \hat{W} = \{\hat{\varphi} < 0\} \quad \text{and} \quad \Omega_{out} \cap \hat{W} = \{\varphi > 0\}.$$

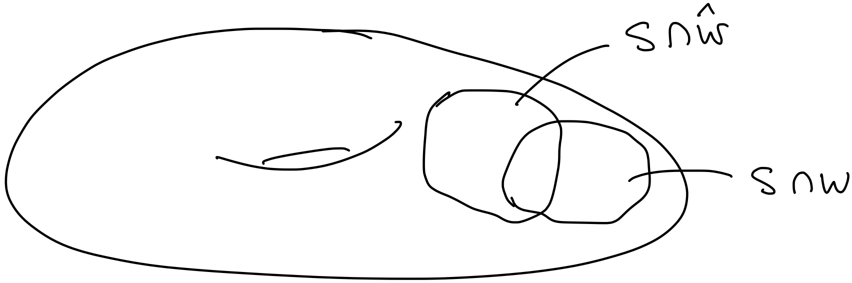
Then $\dfrac{\nabla\hat{\varphi}(p)}{|\nabla\hat{\varphi}(p)|}$ is necessarily pointing inwards. In particular,

$$\frac{\nabla\hat{\varphi}(p)}{|\nabla\hat{\varphi}(p)|} = \frac{\nabla\varphi(p)}{|\nabla\varphi(p)|} \qquad \forall p \in W \cap \hat{W} \cap S.$$

That is

$$n(p) := \begin{cases} \dfrac{\nabla\varphi(p)}{|\nabla\varphi(p)|} & \text{if } p \in S\cap W, \\[2ex] \dfrac{\nabla\hat{\varphi}(p)}{|\nabla\hat{\varphi}(p)|} & \text{if } p \in S\cap \hat{W}, \end{cases}$$

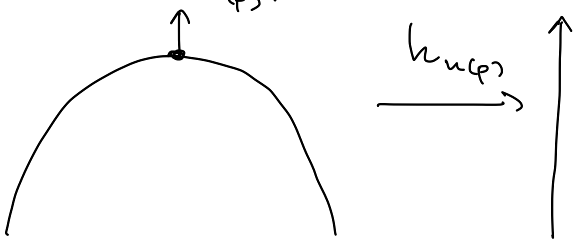
is well-defined and smooth on $S \cap (W \cup \hat{W})$.



Since we can cover all of S by such subsets, n is a well-defined unit normal field pointing outwards. $\square$

Cor Let S be a cmpt surface with positive Gauss curvature. If n is the unit normal field pointing outwards, then the second fundamental form of S with respect to n is positive-definite.

Proof Pick $p \in S$ and consider the height function $h_{n(p)}$. This has a local maximum



at p, hence

$$\mathrm{Hess}\, h_{n(p)} = -\mathrm{I\!I}_p < 0 \iff \mathrm{I\!I}_p > 0. \quad \square$$

Prop Let $S \subset \mathbb{R}^3$ be a cmpt connected surf. If $K(p) > 0$ $\forall p \in S$, then Ω_{in} is convex, that is

$$x, y \in \Omega_{in} \implies [x, y] \subset \Omega_{in}.$$

the segment in $\mathbb{R}^3$ connecting x and y.

In particular, $\overline{\Omega_{in}}$ is also convex and

$$x, y \in S \implies]x, y[\subset \Omega_{in}.$$

Proof Assume $\Omega = \Omega_{in}$ is not convex. Consider

$$A := \{ (x,y) \in \Omega \times \Omega \mid [x,y] \subset \Omega \}.$$

Notice that

- $A \neq \emptyset$, since $(x,x) \in A \quad \forall x \in \Omega$
- $A \neq \Omega \times \Omega$, since otherwise Ω were convex.

Then the topological boundary ∂A of $A \subset \Omega \times \Omega$ is non-empty. This means the following:

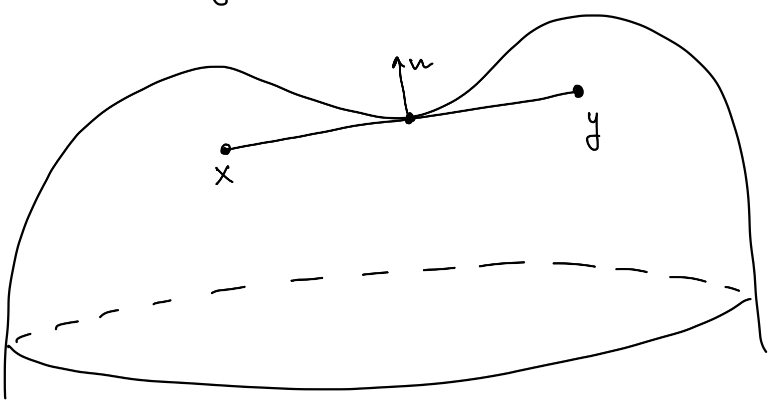
$\exists$ sequences $x_n, y_n, x_n', y_n' \in \Omega$ s.t.

$$x_n, x_n' \longrightarrow x \in \Omega, \quad y_n, y_n' \longrightarrow y \in \Omega$$

$$[x_n, y_n] \subset \Omega \quad \text{and} \quad [x_n', y_n'] \not\subset \Omega$$

<u>Exercise</u> Show that $\exists\, z \in [x,y] \cap \underbrace{\partial\Omega}_{= S}$

s.t. $v := y - x \in T_z S$.



This yields: $[x,y] \subset T_z S$.

Let n be a unit normal vector at z pointing outwards (locally, so that a nbhd of z in S is located below the tangent plane). Then $\mathrm{Hess}_z h_n < 0$ so that h_n has a strict loc. max. at z.

Furthermore, can assume $z=0$, $n=(0,0,1)$, and $v=(1,0,0)$.

$S = \{ (u,v, f(u,v)) \}$ in a nbhd of the origin.

Consider the curve $\gamma : (-\varepsilon, \varepsilon) \to S$

$\gamma(t) = (t, 0, f(t,0))$.

Since $\gamma(t)$ lies above $(t,0,0)$, we must have $f(t,0) \geq 0$ and $f(0,0)=0$. Hence,

$t=0$ must be a pt of loc. min. for $t \mapsto f(t,0)$. This is impossible, because

$h_n \circ \gamma : t \longmapsto f(t,0)$

must have a strict loc. max. at $t=0$. $\boxtimes$

<u>Solution of the exercise in the pf</u>

Since $[x_n', y_n'] \not\subset \Omega$, there exists $z_n' = t_n x_n' + (1-t_n) y_n' \notin \Omega$ for some $t_n \in [0,1]$.

By the compactness of $[0,1]$, there

there exists a subseq. t_{n_m} converging to some $t \in [0,1]$. In fact, $t \in (0,1)$ since the endpoint of $[x,y]$ belong to Ω by construction.

Furthermore, any neighbourhood of $z := tx + (1-t)y$ contains pts from the complement of Ω, for example z'_{n_m} for m sufficiently large. However, any nbhd of z contains also points from Ω, for example $z_{n_m} := t_{n_m} x_{n_m} + (1-t_{n_m}) y_{n_m}$ provided m is sufficiently large. Hence, $z \in \partial\Omega = S$.

Assume $v \notin T_z S$. Then any nbhd of z in $[x,y]$ would contain pts both from Ω and $\mathbb{R}^3 \setminus \Omega$. Indeed, if S is given by the equation $\varphi(p) = 0$ in a nbhd of z, then $v \notin T_z S \iff \langle \nabla\varphi(z), v \rangle \neq 0$

$$\Rightarrow \quad \varphi(z + tv) = \underbrace{\varphi(z)}_{=0} + t \underbrace{\langle \nabla\varphi(z), v\rangle}_{\neq 0} + o(t^2)$$

$\Rightarrow$ φ takes both positive and negative values on $[z - \varepsilon v, z + \varepsilon v]$. This is impossible, since otherwise $[x_{n_m}, y_{n_m}]$ cannot be contained in Ω.

<u>Prop</u> Let S be a surface with positive Gauss curvature. The affine tangent plane

$$T_p^a S = \{ p + v \mid v \in T_p S \}$$

intersects S at p only.

<u>Proof</u> Assume $q \in T_p^a S \cap S$, $q \neq p$.

$\Longrightarrow$ $]p, q[\in \Omega_{in}$ by the Prop. on P. 16.

However, the positivity of the Gauss curvature implies that all pts in a nbhd of p in $T_p^a S$ lie in Ω_{out}. This is a contradiction $\square$

Thm Let S be a compact connected surface. If $K(p) > 0$ $\forall p \in S$, then the Gauss map of S

$$n : S \longrightarrow S^2$$

is a diffeomorphism.

Proof

Step 1 The Gauss map is a local diffeo.

$K(p) := \det(d_p n) \neq 0 \implies$

$d_p n$ is an iso $\implies$ n is a loc. diffeo

by the inverse function thm.

Step 2 The Gauss map is surjective.

S is cmpt $\implies$ $n(S) \subset S^2$ is cmpt

$\implies$ $n(S)$ is closed, since S^2 is Hausdorff

Also, $n(S)$ is clearly non-empty.

Step 1 $\implies$ $n(S)$ is open $\implies$ $n(S) = S^2$
since S^2 is connected.

Step 3 The Gauss map is injective.

Given $n \in S^2$ consider the height function

$$H_n : \overline{\Omega}_{in} \longrightarrow \mathbb{R}$$
$$x \longmapsto \langle n, x \rangle$$

Notice that $H_n \big|_{\partial\overline{\Omega}_{in} = S} = h_n$.

Notice that any pt of loc. max. of H_n must be on $\partial\overline{\Omega}_{in} = S$, since $\nabla H_n \neq 0$ at any interior pt of $\overline{\Omega}_{in}$.

Assume H_n has two distinct pts of loc. maxima. Denote these pts by p and q. Can assume

$$H_n(p) \geq H_n(q).$$

<u>Case 1.</u> $H_n(p) > H_n(q)$

Then we have

$$H_n(tp + (1-t)q) = t H_n(p) + (1-t) H_n(q)$$
$$> t H_n(q) + (1-t) H_n(q) = H_n(q)$$

For $t \to 0$, $t > 0$ we have

$p_t := tp + (1-t)q \to q$ and $H_n(p_t) > H_n(q)$.

Thus, q cannot be a pt of loc. max. ~~for~~ H_n.

<u>Case 2.</u> $H_n(p) = H_n(q)$

$\iff \langle n, p-q\rangle = 0$

$\implies p-q \in T_p S$

$\implies p + t(p-q) \in T_p^a S \qquad \forall t \in \mathbb{R}$

$\overset{t=-1}{\implies} q \in T_p^a S \implies q = p.$ Contradiction.

This shows that H_n has at most one loc. maximum on $\overline{\Omega}_{in}$. Since $\overline{\Omega}_{in}$ is cmpt, such pt must exist, so that H_n has a unique pt of loc. maximum p, which lies on S.

Then p is also a unique pt of loc. max. for h_n, that is a unique solution of

$$n(q) = n.$$

Thus, Step 2 + Step 3 $\implies \exists$ the inverse to the Gauss map

Step 1 $\implies$ this map is smooth. $\boxtimes$

<u>Corollary</u> Let S be any compact surface with positive Gauss curvature K. Then

$$\int_S K = 4\pi.$$

<u>Proof</u>

$$\int_S K \underset{K>0}{=} \int_S |K| \underset{\text{Defn of } K}{=} \int_S |\det(dn)|$$

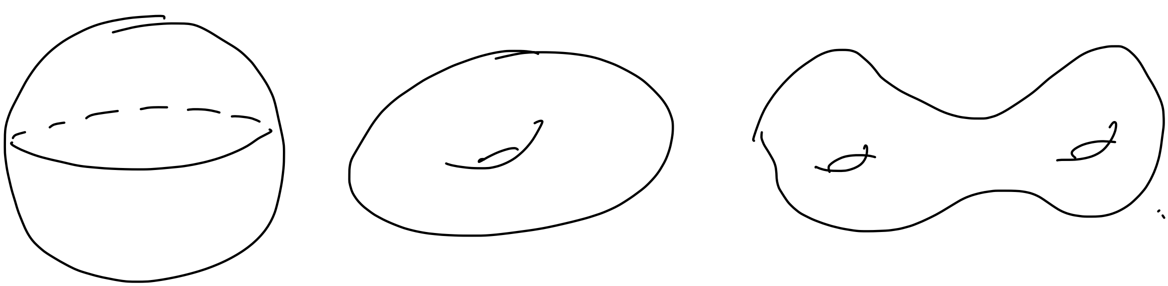
$$\underset{\text{Part 3, Thm on P.15}}{=} \int_{S^2} 1 = \text{Area}(S^2) = 4\pi$$

$\square$

<u>Rem</u> It turns out that only our proof requires $K>0$, however for any S diffeomorphic to S^2 we have

$$\int_S K = 4\pi.$$

Even more generally, let g denote the number of "holes" of S :



$g=0$ $\qquad$ $g=1$ $\qquad$ $g=2$ $\qquad$ ...

Then we have

$$\boxed{\int_S K = 4\pi(1-g)}$$

for any cmpt surface. This is the celebrated Gauss - Bonnet thm.