Manifolds

There are a number of ways we could generalize our discussion of surfaces.

1. Hypersurfaces These are subsets $S \subset \mathbb{R}^{k+1}$ admitting (smooth) parametrizations $\varphi: V \to U \subset S$, $V \subset \mathbb{R}^k$ in a nbhd of each pt just as in the definition of the surface. Virtually all notions and theorems about surfaces generalize immediately to this case (together with proofs), since the condition $k=2$ was never used in an essential way.
2. Embedded submanifolds. Roughly speaking these are "k-dimensional surfaces in $\mathbb{R}^{k+\ell}$". More formally, we could call $S \subset \mathbb{R}^{k+\ell}$ a k-dimensional submanifold, if S admits parametrizations $\varphi: V \to U \subset S$, where $V \subset \mathbb{R}^k$ is open and $D\varphi$ is injective at each pt. Most of the statements about surfaces we have seen generalize to this case too (and rather trivially) except the very last section involving the Gauss map. It generalizes too, however, this requires some extra work

and, more importantly, not all statements made for surfaces hold true in this case.

Abstract manifolds

Nonetheless, even higher degree of abstraction is required for applications. Hence, we adapt the following approach.

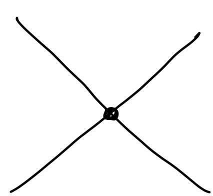
Def A Hausdorff topological space M is said to be a topological manifold of dimension k, $k \in \mathbb{N}_0$, if M is locally homeomorphic to $\mathbb{R}^k$.

To explain: $\forall m \in M$ $\exists$ a nbhd $U \subset M$ and a homeomorphism $\varphi : U \to V \subset \mathbb{R}^k$, where V is open. A pair (U, φ) is called <u>a chart</u> on M.

Ex 1) Any surface is a topological mfld of dimension $k=2$.

2) S^k is a topological mfld of dimension n. This can be seen by covering S^k by two charts just in the case $k=2$.

<u>A non-example</u>: $\{(x,y) \in \mathbb{R}^2 \mid x^2 = y^2\}$ is not a topological mfld.



<u>Def</u>. A collection $\mathcal{U} = \{(U_\alpha, \varphi_\alpha) \mid \alpha \in A\}$ of charts on M is called a C^0-atlas, if

$$\bigcup_{\alpha \in A} U_\alpha = M.$$

- A C^0-atlas is called smooth, if each coordinate transformation

$$\theta_{\alpha\beta} := \varphi_\alpha \circ \varphi_\beta^{-1} : \underset{\mathbb{R}^k}{\underset{\cap}{\varphi_\beta(U_\alpha \cap U_\beta)}} \longrightarrow \underset{\mathbb{R}^k}{\underset{\cap}{\varphi_\alpha(U_\alpha \cap U_\beta)}}$$

is smooth.

- A smooth manifold is a topological mfld together with a smooth atlas.

<u>Examples</u>

0) $\mathbb{R}^k$ with a single chart $(\mathbb{R}^k, id)$ is a smooth mfld.
More generally, any open subset of $\mathbb{R}^k$ is a smooth mfld

1) Any surface is a smooth mfld of dim. = 2.

2) S^k is a smooth mfld of dim k.

3) $\mathbb{R}P^k$ = the set of all lines in $\mathbb{R}^{k+1}$ through the origin

$$= \mathbb{R}^{k+1} \setminus \{0\} / \sim \qquad (x_0, \dots, x_k) \sim (\lambda x_0, \dots, \lambda x_k), \ \lambda \in \mathbb{R} \setminus \{0\}.$$

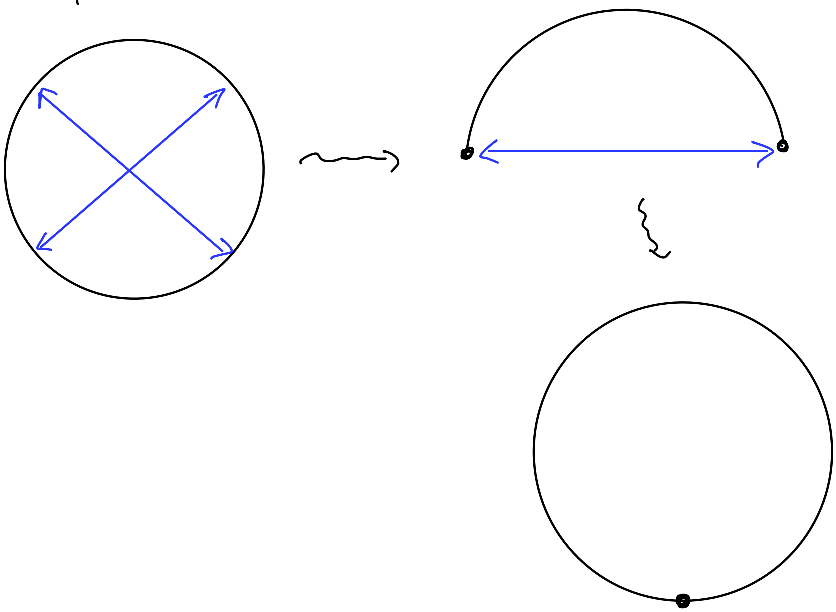
$$= S^k / \sim \qquad x \sim -x$$

Define the topology on $\mathbb{R}P^k$ as the quotient topology of $\mathbb{R}^{k+1}\setminus\{0\}$, that is

$U \subset \mathbb{RP}^k$ is open $\iff$ $\pi^{-1}(U) \subset \mathbb{R}^{k+1}\setminus\{0\}$ is open

where $\pi : \mathbb{R}^{k+1}\setminus\{0\} \to \mathbb{R}^{k+1}\setminus\{0\}/\sim \; = \mathbb{RP}^k$ is the quotient map.

For example, if $k=1$ $\mathbb{RP}^1$ is homeomorphic to S^1:



Define

$$U_j := \{ [x_0 : \dots : x_n] \in \mathbb{RP}^k \mid x_j \neq 0 \} \quad j = 0, \dots, k$$

U_j is open, since

$$\pi^{-1}(U_j) = \{ (x_0, \dots, x_k) \in \mathbb{R}^{k+1}\setminus\{0\} \mid x_j \neq 0 \} \text{ is open.}$$

Consider the map $\varphi_j : U_j \to \mathbb{R}^k$

$$\varphi_j([x]) = \left(\frac{x_0}{x_j}, \dots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \dots, \frac{x_k}{x_j} \right)$$

Then the map $\psi_j : \mathbb{R}^n \to U_j$

$$\psi_j(y_0, \dots, y_{k-1}) = [y_0 : \dots : y_{j-1} : 1 : y_j : \dots : y_{k-1}]$$

is the inverse of φ_j. In particular, φ_j is a homeomorphism. Thus,

$$\mathcal{U} = \{ (U_j, \varphi_j) \mid j = 0, \dots, k \}$$

is a C^0-atlas on $\mathbb{RP}^k$.

Consider the coordinate transformation

$$\theta_{01} = \varphi_0 \circ \varphi_1^{-1} = \varphi_0 \circ \psi_1$$

$$\theta_{01}(y_0, \dots, y_{k-1}) = \varphi_0([y_0 : 1 : y_1 : \dots : y_{k-1}])$$

$$= \left(\frac{1}{y_0}, \frac{y_1}{y_0}, \frac{y_2}{y_0}, \dots, \frac{y_{k-1}}{y_0} \right)$$

which is smooth on

$$\{ y \in \mathbb{R}^k \mid y_0 \neq 0 \} = \varphi_0(U_0 \cap U_1)$$

The reader should check that in fact each $\theta_{ij} = \varphi_i \circ \varphi_j^{-1}$ is smooth. Thus, $\mathcal{U}$ is in fact a smooth atlas.

Rem For $k=2$ we obtain a smooth mfld of dimension 2, however it turns out that $\mathbb{R}P^2$ cannot be represented as a surface in $\mathbb{R}^3$. We would have discovered this mfld if we would consider non-orientable surfaces more carefully. Indeed, the Gauss map of a non-orientable surface $S \subset \mathbb{R}^3$ is naturally defined as a map

$$S \ni p \longmapsto (T_p S)^\perp \in \mathbb{R}P^2.$$

4) Products If M and N are smooth manifolds of dimensions k and ℓ respectively, then $M \times N$ is a smooth mfld of dimension $k+\ell$. Indeed, if $\mathcal{U} = \{(U_\alpha, \varphi_\alpha) \mid \alpha \in A\}$ is a smooth atlas on M and $\mathcal{V} = \{(V_\beta, \xi_\beta) \mid \beta \in B\}$ is a smooth atlas on N, that

$$\mathcal{W} := \{(U_\alpha \times V_\beta, \varphi_\alpha \times \xi_\beta) \mid \alpha \in A, \beta \in B\}$$

is a smooth atlas on $M \times N$.

Exercise Find the coordinate transformations of the atlas $\mathcal{W}$ and show that these are smooth indeed.

In particular,

(i) $\mathbb{T}^k = S^1 \times \dots \times S^1$ is a smooth mfld of dimension k

(ii) the cylinder $\mathbb{R} \times S^1$ is a smooth mfld of dimension 2.

A smooth atlas does not need to be unique. For example, on S^2 we can choose

$$\mathcal{U} = \{ (S^2 \setminus \{N\}, \varphi_N), (S^2 \setminus \{S\}, \varphi_S) \}$$

or the atlas consisting of 6 hemispheres. This leads to the following definition.

<u>Def</u> Two atlases $\mathcal{U} = \{ (U_\alpha, \varphi_\alpha) \mid \alpha \in A \}$ and $\mathcal{V} = \{ (V_\beta, \xi_\beta) \mid \beta \in B \}$ on the same top. space M are said to be equivalent if $\mathcal{U} \cup \mathcal{V}$ is also a smooth atlas, that is if

$$\xi_\beta \circ \varphi_\alpha^{-1} \quad \text{and} \quad \varphi_\alpha \circ \xi_\beta^{-1}$$

are smooth $\forall \alpha \in A$ and $\forall \beta \in B$.

<u>Def</u> A smooth structure is an equivalence class of atlasses.

<u>Rem</u> In what follows we shall feel free to replace an atlas by an equivalent one.

Smooth maps

Let $(M, \mathcal{U})$ be a smooth mfld.

Def A function $f : M \to \mathbb{R}$ is called smooth, if $\forall \alpha \in A$

$$F_\alpha := f \circ \varphi_\alpha^{-1} : \mathbb{R}^k \longrightarrow \mathbb{R}$$

is smooth.

Just as in the case of surfaces, F_α is called the coordinate representation of f with respect to the chart $(U_\alpha, \varphi_\alpha)$.

Also, just as in the case of surfaces each F_α is defined on an open subset of $\mathbb{R}^k$, namely $\varphi_\alpha(U_\alpha)$. This should be clear by now and will not be mentioned explicitly below unless really necessary.

Exercise Let $f : M \to \mathbb{R}$ be a function. If $\mathcal{U} \sim \mathcal{V}$, show that

f is smooth wrt $\mathcal{U} \iff f$ is smooth wrt $\mathcal{V}$.

Prop The set $C^\infty(M)$ of all smooth functions on a smooth mfld is an algebra, that is

- $\left.\begin{array}{l} f, g \in C^\infty(M) \\ \lambda, \mu \in \mathbb{R} \end{array}\right\} \Rightarrow \Big\{ \lambda f + \mu g \in C^\infty(M)$

- $f, g \in C^\infty(M) \Rightarrow f \cdot g \in C^\infty(M)$ ▨

The proof of this proposition is similar to the proof of the analogous proposition for the surfaces.

More generally, let $(M, \mathcal{U})$ and $(N, \mathcal{V})$ be two smooth mflds of dimensions k and ℓ respectively.

Def A map $f: M \to N$ is said to be smooth, if each coordinate representation

$$\xi_\beta \circ f \circ \varphi_\alpha^{-1} : \mathbb{R}^k \longrightarrow \mathbb{R}^\ell$$

is smooth.

Prop

$$\left.\begin{array}{l} f \in C^\infty(M; N) \\ g \in C^\infty(N; L) \end{array}\right\} \Rightarrow \Big\{ g \circ f \in C^\infty(M; L)$$ ▨

Again, the proof of this proposition is a verbatim repetition of the corresp. proof for the surfaces.

Also, just in the case of surfaces, we have the notions of a diffeomorphism and a local diffeomorphism.

The tangent space

If M is an abstract mfld and $\gamma : (-\varepsilon, \varepsilon) \longrightarrow M$ is a smooth curve, then $\dot{\gamma}(0)$ does not make sense in any obvious way. Hence, our definition of the tangent space does not immediately generalize to the present setting.

To come up with a suitable generalization, observe the following: $v \in \mathbb{R}^k$ is the tangent vector of a curve $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^k$, $\gamma(0) = p$, if and only if

$$\gamma(t) = p + v \cdot t + o(t) \qquad \text{as } t \to 0.$$

Hence we may consider the following equivalence relation: two smooth curves $\gamma_1, \gamma_2 : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}^k$, $\gamma_1(0) = p = \gamma_2(0)$, are said to be <u>equivalent</u> if

$$\gamma_1(t) - \gamma_2(t) = o(t).$$

Our observation above yields immediately.

<u>Prop</u>

$$\gamma_1 \sim \gamma_2 \iff \dot{\gamma}_1(0) = \dot{\gamma}_2(0).$$ ▨

This equivalence relation makes sense on manifolds as follows.

<u>Def</u> Let M be a smooth mfld of dimension k. Pick a pt $m \in M$. Two smooth curves $\gamma_1, \gamma_2 : (-\varepsilon, \varepsilon) \to M$ s.t. $\gamma_1(0) = m = \gamma_2(0)$ are said to be equivalent, if for any chart (U, φ) s.t. $m \in U$ we have

$$\varphi \circ \gamma_1 \sim_{\mathbb{R}^k} \varphi \circ \gamma_2 \iff \varphi \circ \gamma_1(t) - \varphi \circ \gamma_2(t) = o(t) \quad (*)$$

An equivalence class of curves is called a tangent vector at the pt m.

<u>Lemma</u> If $(*)$ holds for <u>some</u> chart (U, φ) s.t. $m \in U$, then $(*)$ holds for <u>any</u> chart containing m.

Proof Let $(\hat{U}, \hat{\varphi})$ be any other chart s.t. $m \in \hat{U}$. Denote $p := \varphi(m)$. Then

$$\hat{\varphi} \circ \gamma_1(t) - \hat{\varphi} \circ \gamma_2(t) = \underbrace{\hat{\varphi} \circ \varphi^{-1}}_{\Theta} \circ \underbrace{\varphi \circ \gamma_1}_{\beta_1}(t) - \underbrace{\hat{\varphi} \circ \varphi^{-1}}_{\Theta} \circ \underbrace{\varphi \circ \gamma_2}_{\beta_2}(t)$$

$$= \Theta \circ \beta_1(t) - \Theta \circ \beta_2(t)$$

Since Θ is smooth, Θ is Lipschitz, that is $\exists\, L > 0$ s.t.

$$|\Theta(x) - \Theta(y)| \leq L\, |x - y| \qquad \forall\, x, y \in B_\delta(p)$$

where $\delta > 0$.

Hence,

$$|\Theta \circ \beta_1(t) - \Theta \circ \beta_2(t)| \leq L\, \underbrace{|\beta_1(t) - \beta_2(t)|}_{= O(t)} = O(t).$$

$\blacksquare$

Def $T_m M := \{ [\gamma] \mid \gamma \text{ is a smooth curve through } m \}$ is called the tangent space of M at m.

Prop / Defn $T_m M$ is a vector space of dim k.

Proof Pick a chart (U, φ) containing m and denote $\varphi(m) = p \in \mathbb{R}^k$. Consider the map

$$\{\gamma \mid \gamma(0) = m\} \longrightarrow \mathbb{R}^k \qquad (*)$$

smooth curve in M.

$$\gamma \longmapsto \frac{d}{dt}\Big|_{t=0} \underbrace{\varphi \circ \gamma(t)}_{\text{curve in } \mathbb{R}^k}.$$

<u>Exercise</u> Show that this map is surjective.

If $\gamma_1 \sim \gamma_2$, then $\beta_1 := \varphi \circ \gamma_1 \sim \beta_2 := \varphi \circ \gamma_2$ by the proof of the lemma on P. 11. Hence, $\dot{\beta}_1(0) = \dot{\beta}_2(0)$ so that we have a well-defined surjective map

$$\varphi_* : T_m M \longrightarrow \mathbb{R}^k.$$

In fact, φ_* is bijective:

$$\varphi_*[\gamma_1] = \varphi_*[\gamma_2] \iff \beta_1 \sim \beta_2 \iff \dot{\beta}_1(0) = \dot{\beta}_2(0)$$

Thus, we define the structure of a vector space on $T_m M$ so that φ_* is a linear isomorphism.

<u>Exercise</u> Show that the following holds:

(i) If $\gamma \in T_m M$ and $\lambda \in \mathbb{R}$, then

$$\lambda[\gamma] = [\gamma(\lambda\,\cdot)]$$

(ii) For two curves γ_1, γ_2 through m define

$$\gamma(t) := \varphi^{-1}\big(\beta_1(t) + \beta_2(t)\big),$$

where $\beta_j := \varphi \circ \gamma_j$. Show that γ is a smooth curve through m and

$$[\gamma_1] + [\gamma_2] = [\gamma].$$

We still need to show that the structure of the vector space on $T_m M$ does not depend on the choice of the chart (U, φ). To this end, let $(\hat{U}, \hat{\varphi})$ be another chart s.t. $m \in \hat{U}$. Let

$$\hat{\varphi}_* : T_m M \longrightarrow \mathbb{R}^k, \quad [\gamma] \longmapsto \frac{d}{dt}\Big|_{t=0} \hat{\varphi} \circ \gamma(t)$$

be the corresponding map. Denoting temporarily by $+_\varphi$ the addition obtaine via φ_*, we obtain

$$[\gamma] = [\gamma_1] +_\varphi [\gamma_2] \iff \dot{\beta}(0) = \dot{\beta}_1(0) + \dot{\beta}_2(0)$$

where $\beta(t) = \varphi \circ \gamma(t)$ and $\beta_j(t) = \varphi \circ \gamma_j(t)$.

Denote $\hat{\beta}(t) = \hat{\varphi} \circ \gamma(t)$ and $\hat{\beta}_j(t) = \hat{\varphi} \circ \gamma_j(t)$.

Then

$$\hat{\beta} = \hat{\varphi} \circ \gamma = \hat{\varphi} \circ \varphi^{-1} \circ \varphi \circ \gamma = \Theta \circ \beta \implies$$

$$\dot{\hat{\beta}}(0) = D_p\Theta(\dot{\beta}(0)).$$

Similarly, we have $\dot{\hat{\beta}}_j(0) = D_p\Theta(\dot{\beta}_j(0))$.

Since $D_p\Theta$ is a linear map, we have

$$\begin{aligned}\dot{\hat{\beta}}(0) &= D_p\Theta(\dot{\beta}_1(0) + \dot{\beta}_2(0)) \\ &= D_p\Theta(\dot{\beta}_1(0)) + D_p\Theta(\dot{\beta}_2(0)) \\ &= \dot{\hat{\beta}}_1(0) + \dot{\hat{\beta}}_2(0)\end{aligned}$$

$$\iff \underset{\substack{\| \\ [\gamma_1] +_{\varphi} [\gamma_2]}}{[\gamma]} = [\gamma_1] +_{\hat{\varphi}} [\gamma_2]$$

$[\gamma_1] +_{\varphi} [\gamma_2]$.

The fact that the multiplication with scalars is independent of the choice of a chart follows immediately from Part (i) of the exercise on P. 13. $\boxed{/\!/}$

<u>Rem</u> The origin in $T_m M$ is represented by the constant curve $\gamma(t) = m \ \forall t$ (or any other curve equivalent to this one).

Prop If $S \subset \mathbb{R}^3$ is a smooth surface, then T_pS in the sense of the definition on P. 12 is naturally isomorphic to the tangent plane of S.

Proof Denote temporarily the tangent plane of S at p by E_p. Consider the map

$$\begin{aligned} T_pS &\longrightarrow E_p \\ [\gamma] &\longmapsto \dot\gamma(0) \end{aligned} \qquad (*)$$

Exercise Check that this map is well-defined, that is is independent of the choice of the representative.

This map is linear. Indeed, if φ is a parametrization at p such that $\varphi(0)=p$ and

$$\gamma_1 := \varphi \circ \beta_1, \qquad \gamma_2 := \varphi \circ \beta_2$$

then $[\gamma_1] + [\gamma_2]$ is represented by the curve $t \mapsto \varphi(\beta_1(t) + \beta_2(t)) = \gamma(t)$

Hence

$$\dot\gamma(0) = D_0\varphi\big(\dot\beta_1(t) + \dot\beta_2(t)\big)$$

$$= D_0\varphi\big(\dot\beta_1(t)\big) + D_0\varphi\big(\dot\beta_2(0)\big)$$

$$= \dot\gamma_1(0) + \dot\gamma_2(0).$$

That is $[\gamma_1]+[\gamma_2] \in T_pS$ is mapped onto $\dot{\gamma}_1(0)+\dot{\gamma}_2(0)$.

Since $\lambda[\gamma] = [\gamma(\lambda\cdot)] \longmapsto \frac{d}{dt}\Big|_{t=0} \gamma(\lambda t) =$

$= \lambda \dot{\gamma}(0)$, we see that (14.*) is linear. Since this map is clearly surjective and T_pS and E_p have equal dimension, (14.*) is an isomorphism. ▨

<u>Rem</u> The proof of the prop/defn on P. 12 implies the following. Let (U, φ) be a chart on M and $m \in U$. Denote $\varphi(m) = p \in \mathbb{R}^k$ and define $\gamma_j : (-\varepsilon, \varepsilon) \to U$, $\gamma_j(0) = m$ by

$$\varphi \circ \gamma_j(t) = p + (0, \dots, 0, \underset{j\text{th place}}{t}, 0, \dots, 0).$$

Then

$$e_\varphi := ([\gamma_1], \dots, [\gamma_k])$$

is a basis of T_mM.

The differential of a smooth map.

Let $f: M^k \to N^\ell$ be a smooth map.

Def For $m \in M$ the map

$$d_m f : T_m M \to T_{f(m)} N$$

$$[\gamma] \mapsto [f \circ \gamma]$$

is called the differential of f at m.

Prop The differential is a linear map.
If (U, φ) is a chart on M s.t. $m \in U$ and (V, ψ) is a chart on N s.t. $f(m) \in V$, then $d_m f$ is represented by the Jacobi matrix of $F := \psi \circ f \circ \varphi^{-1}$ with respect to the bases e_φ and e_ψ, that is

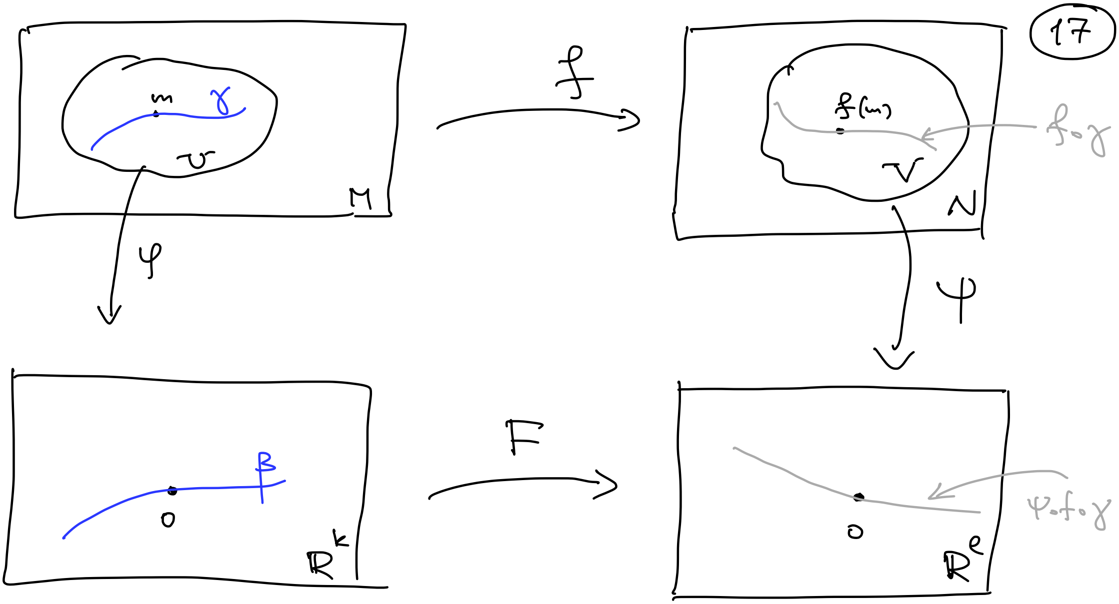
$$d_m f \,(e_\varphi) = e_\psi \cdot D_{\varphi(m)} F$$

Proof Assume for simplicity $\varphi(m) = 0 \in \mathbb{R}^k$

$\psi(f(m)) = 0 \in \mathbb{R}^\ell$

For $\gamma : (-\varepsilon, \varepsilon) \to M$ s.t. $\gamma(0) = m$, denote

$$\beta := \varphi \circ \gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^k$$



We have

$$f\circ\gamma = f\circ\varphi^{-1}\circ\underbrace{\varphi\circ\gamma}_{\beta} = f\circ\varphi^{-1}\circ\beta.$$

Then

$$\psi_*[f\circ\gamma] = \frac{d}{dt}\Big|_{t=0}\Big(\underbrace{\psi\circ f\circ\varphi^{-1}}_{F}\circ\beta(t)\Big)$$

$$= \frac{d}{dt}\Big|_{t=0}\big(F\circ\beta(t)\big)$$

$$= D_0F\,(\dot\beta(0)).$$

However, $\dot\beta(0) = \frac{d}{dt}\Big|_{t=0}\varphi\circ\gamma(t) = \varphi_*[\gamma]$

Hence,

$$\psi_* \, d_m f([\gamma]) = D_0 F \circ \varphi_* [\gamma] \qquad \forall \gamma$$

$$\Longleftrightarrow \quad d_m f = \underbrace{\psi_*^{-1} \circ D_0 F \circ \varphi_*}_{\text{linear}} \quad \Longrightarrow \quad d_m f \text{ is linear}$$

Furthermore, by the definition of $e_\varphi = ([\gamma_1], \dots, [\gamma_k])$, we have

$$\dot\beta_j(0) = (0, \dots, 0, \underset{j\text{th place}}{1}, 0, \dots, 0) = e_j$$

$$\psi_* \, d_m f([\gamma_j]) = D_0 F(e_j) = \sum_{i=1}^{\ell} \frac{\partial F_i}{\partial x_j} \hat{e}_i,$$

where $(\hat{e}_1, \dots, \hat{e}_\ell)$ is a standard basis of $\mathbb{R}^\ell$.

Hence,

$$d_m f [\gamma_j] = \psi_*^{-1} \left(\sum_{i=1}^{\ell} \frac{\partial F_i}{\partial x_j} \hat{e}_i \right)$$

$$= \sum_{i=1}^{\ell} \frac{\partial F_i}{\partial x_j} \underbrace{\psi_*^{-1}(\hat{e}_i)}_{[\hat\gamma_i]}$$

ith element of e_ψ

$$\Longrightarrow (d_m f[\gamma_1], \dots, d_m f[\gamma_k]) = ([\hat\gamma_1], \dots, [\hat\gamma_\ell]) \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_k} \\ \vdots & & \vdots \\ \frac{\partial F_\ell}{\partial x_1} & \cdots & \frac{\partial F_\ell}{\partial x_k} \end{pmatrix}$$

$\square$

Prop For any smooth mflds M, N, K and any $f \in C^\infty(M;N)$ and $g \in C^\infty(N;K)$ we have

$$d_m(g \circ f) = d_{f(m)} g \circ d_m f.$$

Proof For any $[\gamma] \in T_m M$ we have

$$d_m(g\circ f)[\gamma] = [g \circ f \circ \gamma] = [g \circ (f \circ \gamma)]$$

$$\underset{\text{defn of } d_{f(m)} g}{=} d_{f(m)} g([f\circ\gamma])$$

$$\underset{\text{defn of } d_m f}{=} d_{f(m)} g(d_m f[\gamma]).$$

$\square$

Cor If $f : M \to N$ is a diffeomorphism, then $d_m f : T_m M \to T_{f(m)} N$ is an isomorphism. Conversely, if $d_m f$ is an isomorphism, then f is a local diffeomorphism at m. $\square$

Submanifolds

Think of $\mathbb{R}^{k+\ell}$ as $\mathbb{R}^k \times \mathbb{R}^\ell$, where $k \geq 1, \ell \geq 1$. We have the maps

$$\imath_2 : \mathbb{R}^\ell \longrightarrow \mathbb{R}^{k+\ell}, \quad \imath_2(y) = (0, y)$$

$$\pi_2 : \mathbb{R}^{k+\ell} \longrightarrow \mathbb{R}^\ell, \quad \pi_2(\underset{\in \mathbb{R}^k}{x}, \underset{\in \mathbb{R}^\ell}{y}) = y$$

For any smooth map $f : \mathbb{R}^{k+\ell} \to \mathbb{R}^\ell$ defined in a nbhd U of some $p = (x_0, y_0)$ we obtain a linear map

$$D_y f\big|_p : \mathbb{R}^\ell \xrightarrow{\ i_2\ } \mathbb{R}^{k+\ell} \xrightarrow{\ Df|_p\ } \mathbb{R}^\ell . \qquad (*)$$

This map is called <u>the partial derivative</u> of f with respect to y at the point p.

<u>Ex</u> For $k = 2$ and $\ell = 1$, we have $f : \mathbb{R}^3 \to \mathbb{R}$.
Keeping to our notations above, this means

$$f = f(x_1, x_2, y).$$

Hence, $(*)$ yields

$$D_y f\big|_p(\zeta) = Df\big|_p (0, 0, \zeta) = \frac{\partial f}{\partial y}\Big|_p \cdot \zeta$$

Thus $D_y f$ can be identified with the partial derivative wrt the last variable, hence the name.

In general, thinking of linear maps $\mathbb{R}^{k+\ell} \to \mathbb{R}^\ell$ as $(k+\ell) \times \ell$ - matrices, we obtain

$$Df = \Bigg(\underbrace{\ \ D_x f\ \ }_{k} \Bigg| \underbrace{\ \ D_y f\ \ }_{\ell} \Bigg) \Big\} \ell$$

that is $D_y f$ is obtained from the Jacobi-matrix of f by considering the last ℓ columns only.

Assume for simplicity that $p = 0 \in \mathbb{R}^{k+\ell}$.
The following is known from the analysis course.

<u>Thm</u> (The implicit function thm)
Assume that $D_y f|_p : \mathbb{R}^\ell \to \mathbb{R}^\ell$ is an isomorphism. Then $\exists$ a nbhd V_1 of $0 \in \mathbb{R}^k$ and a nbhd V_2 of $0 \in \mathbb{R}^\ell$ and a unique smooth map $h : V_1 \to V_2$ s.t.

$$f(x,y) = 0 \quad \Longleftrightarrow \quad y = h(x)$$

whenever $(x,y) \in V_1 \times V_2$.

Furthermore, denoting $W := f^{-1}(0) \cap (V_1 \times V_2)$, the map

$$\psi := \pi_1|_W : W \to \mathbb{R}^k, \quad \psi(x,y) = x$$

is a homeomorphism, that is (W, ψ) is a chart on $f^{-1}(0) \cap U$. ////

<u>Ex</u> Assume $k = 2$, $\ell = 1$. Then

$$D_y f|_p : \mathbb{R} \to \mathbb{R} \text{ is iso} \iff \frac{\partial f}{\partial y}\Big|_p \neq 0.$$

Hence, locally the eqn $f(x_1, x_2, y) = 0$ can be solved wrt y: $y = h(x_1, x_2)$. Moreover, the map

$$\varphi : \mathbb{R}^3 \to \mathbb{R}^3, \quad \varphi(x_1, x_2, y) = (x_1, x_2, f(x_1, x_2, y))$$

is a local diffeomorphism at 0, since

$$D\varphi|_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ * & * & \frac{\partial f}{\partial y}\Big|_0 \end{pmatrix} \Rightarrow \det(D\varphi|_0) \neq 0.$$

Hence, $\exists$ a nbhd U of $0 \in \mathbb{R}^3$ s.t. (U, φ) can be viewed as a chart on $\mathbb{R}^3$ (more formally, this means that the standard atlas $\{(\mathbb{R}^3, id)\}$ is equivalent to $\{(\mathbb{R}^3, id), (U, \varphi)\}$.)

This chart has the following nice property:

$$\varphi(f^{-1}(0) \cap U) \subset \mathbb{R}^2 \times \{0\}.$$

<u>Exercise</u> Show that in the situation of the implicit function theorem, there exists a nbhd U of $0 \in \mathbb{R}^{k+l}$ and a diffeomorphism $\varphi : U \to \hat{U} \subset \mathbb{R}^{k+l}$ s.t.

$$\varphi(f^{-1}(0) \cap U) \subset \mathbb{R}^k \times \{0\}.$$

This motivates the following definitions.

<u>Def</u> Let N be a smooth manifold with an atlas $\mathcal{U} = \{(U_\alpha, \varphi_\alpha)\}$. A chart (U, φ) is said to be smoothy compatible with $\mathcal{U}$ if $\mathcal{U} \cup \{(U, \varphi)\}$ is a smooth atlas on M.

This means simply that $\varphi: U \to \varphi(U)$ is a diffeomorphism.

<u>Def</u> Let N be a smooth mfld of dimension $k+\ell$. A subset $M \subset N$ is said to be <u>a submanifold</u> of dimension k if for any $m \in M$ there exists a compatible chart (U, φ) such that

$$\varphi(M \cap U) \subset \mathbb{R}^k \times \{0\} \subset \mathbb{R}^{k+\ell}. \qquad (*)$$

A chart (U, φ) as above is called <u>adapted</u> (to M).

It is clear from the discussion above that any surface is a submanifold of $\mathbb{R}^3$ of dimension 2.

<u>Prop</u> A k-submanifold is itself a manifold of dimension k.

<u>Proof</u> A submanifold $M^k \subset N^{k+\ell}$ is equipped with a C^0-atlas $\mathcal{U}$ consisting of all adapted charts.

In fact, $\mathcal{U}$ is smooth. Indeed, let (U_1, φ_1) and (U_2, φ_2) be two charts adapted to M. Denote

$$\imath_1 : \mathbb{R}^k \to \mathbb{R}^{k+\ell}, \quad \imath_1(x) = (x, 0),$$

$$\pi_1 : \mathbb{R}^{k+\ell} \to \mathbb{R}^k, \quad \pi_1(x, y) = x \in \mathbb{R}^k.$$

Then for $\psi_j := \pi_1 \circ \varphi_j$ we have

$$\underbrace{\psi_1 \circ \psi_2^{-1}(x)}_{= \Theta_{12}^M} = \psi_1(\varphi_2^{-1}(x,0)) = \pi_1 \circ \underbrace{\varphi_1 \circ \varphi_2^{-1}}_{\Theta_{12}^N} \circ \imath_1(x)$$

Hence, Θ_{12}^M is smooth as a composition of smooth maps. ▯

<u>Exercise</u> Let $M \subset N$ be a submanifold. Then for any $f \in C^\infty(N)$ the function $f|_M$ is smooth on M.

<u>Def</u> Let $f \in C^\infty(M; N)$. A pt $n \in N$ is said to be a regular value of f, if $\forall m \in f^{-1}(n)$ $d_m f : T_m M \to T_n N$ is surjective.

<u>Rem</u> 1) Any $n \notin \operatorname{Im} f$ is a regular value of f.
2) If f has a reg. value n s.t. $f^{-1}(n) \neq \emptyset$, then $\dim M \geq \dim N$.

The following is one of the main theorems of the course.

<u>Thm</u> If n is a regular value of some $f \in C^\infty(M; N)$ such that $f^{-1}(n) \neq \emptyset$, then $f^{-1}(n) \subset M$ is a smooth manifold of dimension $k := \dim M - \dim N$.

Proof Denote $\ell = \dim N \Rightarrow \dim M = k+\ell$.
Pick any $m \in f^{-1}(n)$ and any charts (U, φ) and (V, ψ) on M and N respectively such that $\varphi(m) = 0$ and $\psi(n) = 0$. Let $F = \psi \circ f \circ \varphi^{-1} : \mathbb{R}^{k+\ell} \to \mathbb{R}^\ell$ be the coordinate representation of f. We have

$$F \circ \varphi = \psi \circ f \Rightarrow \underbrace{d_m(F\circ\varphi)}_{\| \atop d_0 F \circ d_m \varphi} = \underbrace{d_m(\psi \circ f)}_{\| \atop d_n\psi \circ d_m f}$$

$d_m\varphi$ and $d_n\psi$ are isomorphisms by the corollary on P. 19. Since $d_m f$ is surjective, it follows that

$$d_0 F = d_n \psi \circ d_m f \circ (d_m\varphi)^{-1} : \mathbb{R}^{k+\ell} \to \mathbb{R}^\ell$$

is also surjective. In particular,

$$\underbrace{\dim \operatorname{Im}(d_0F)}_{\| \atop \ell} = \underbrace{\dim \mathbb{R}^{k+\ell}}_{\| \atop k+\ell} - \dim \operatorname{Ker}(d_0 F)$$

$$\Rightarrow \boxed{\dim \operatorname{Ker}(d_0 F) = k}$$

Choose a basis $(v_1, \dots, v_{k+\ell})$ of $\mathbb{R}^{k+\ell}$ s.t. $(v_1, \dots, v_k)$ is a basis of $\operatorname{Ker} d_0 F$ and define $A: \mathbb{R}^{k+\ell} \to \mathbb{R}^{k+\ell}$ by $Az = \sum_{j=1}^{k+\ell} z_j v_j$.

Then the following holds:

- A is an iso since it maps the standard basis of $\mathbb{R}^{k+\ell}$ onto $(v_1, \dots, v_{k+\ell})$.
- $A \circ \iota_1 : \mathbb{R}^k \to \operatorname{Ker} d_0 F$ is an iso.
- $d_0 F \circ A \circ \iota_2 : \mathbb{R}^\ell \to \mathbb{R}^\ell$ is an iso.

Furthermore, for $G := F \circ A : \mathbb{R}^{k+\ell} \to \mathbb{R}^\ell$ we have

$$d_0 G = d_0 F \circ d_0 A \underset{\text{A is linear}}{=} d_0 F \circ A$$

$$\Longrightarrow \quad D_y G\big|_0 = d_0 F \circ A \circ \iota_2 \quad \text{is an iso}$$

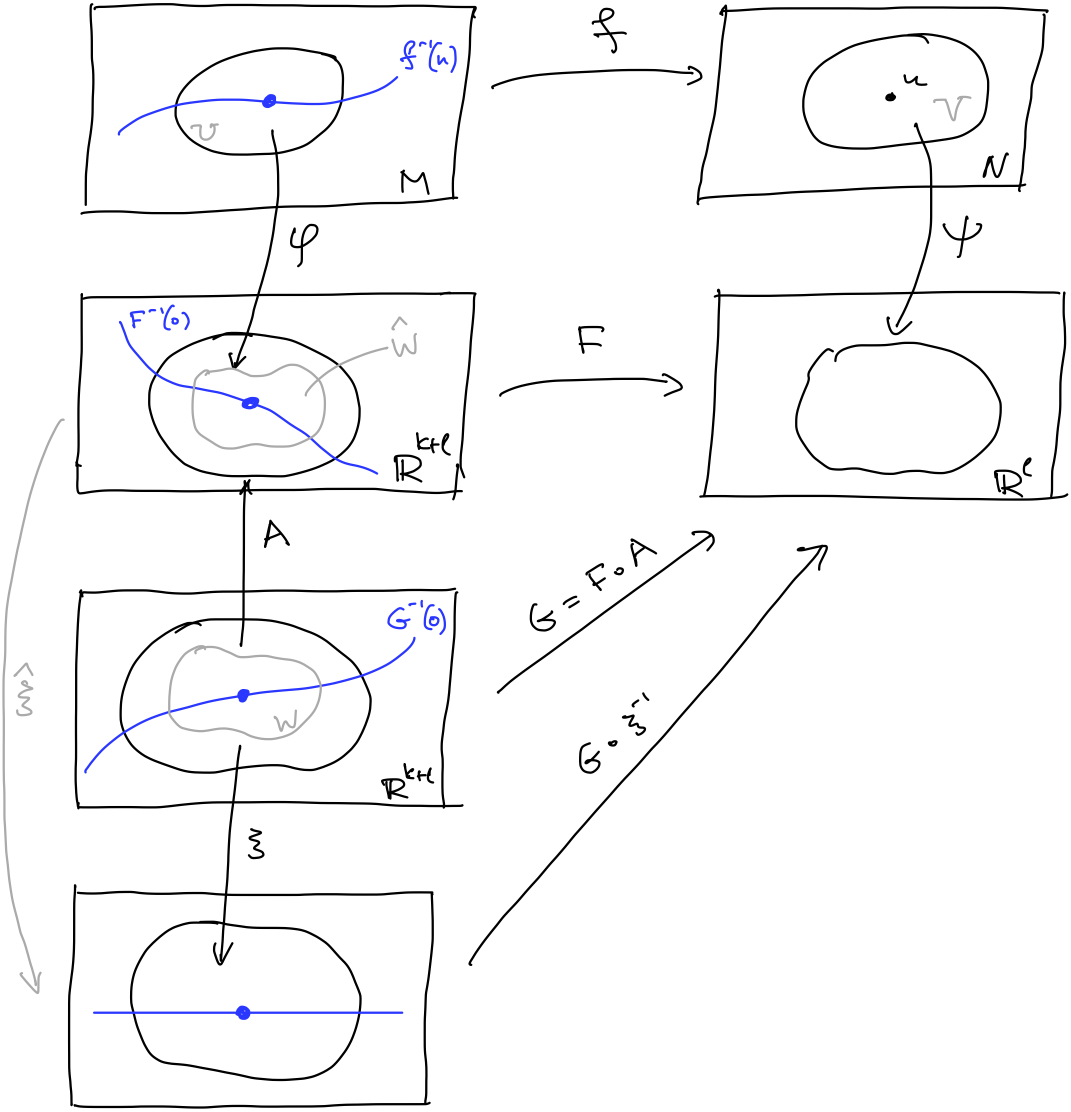
$$\xRightarrow[\text{P. 22}]{\text{Exerc. on}} \quad \exists \text{ a chart } (W, \xi) \text{ adapted to } G^{-1}(0),$$

that is

$$\xi(G^{-1}(0) \cap W) \subset \mathbb{R}^k \times \{0\}.$$

Without loss of generality, we can assume

$$W \subset A^{-1}(\varphi(U)).$$



Define a chart $(\hat{W}, \hat{\xi})$ on $\mathbb{R}^{k+\ell}$ by

$$(\hat{W}, \hat{\xi}) = (A(W), \xi \circ A^{-1})$$

Notice $z \in G^{-1}(0) \iff Az \in F^{-1}(0)$. Hence

$$\hat{\xi}(\hat{W} \cap F^{-1}(0)) = \xi(W \cap G^{-1}(0)) \subset \mathbb{R}^k \times \{0\}.$$

Finally, set

$$\varphi_1 := \hat{\xi} \circ \varphi \quad \text{and} \quad U_1 = \varphi^{-1}(\hat{W}) = \varphi_1^{-1}(\hat{\xi}(\hat{W}))$$

Then

$$\varphi_1(U_1 \cap f^{-1}(n)) = \hat{\xi}(\hat{W} \cap F^{-1}(0)) \subset \mathbb{R}^k \times \{0\}.$$

Thus, (U_1, φ_1) is an adapted chart. ☐

Prop In the setting of the above thm,

$$T_m f^{-1}(0) = \operatorname{Ker} d_m f.$$

Proof Suppose $\gamma : (-\varepsilon, \varepsilon) \to f^{-1}(n)$, $\gamma(0) = m$.

Then $f \circ \gamma(t) = n \quad \forall t \implies$

$$d_m f\,[\gamma] = [f \circ \gamma] = 0,$$

where the last equality follows from the remark on P. 14'. Hence,

$$T_m f^{-1}(n) \subset \operatorname{Ker} d_m f.$$

However, both these spaces have equal dimension k, so that we must have

$$T_m f^{-1}(n) = \operatorname{Ker} d_m f.$$

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