0.1 Vector fields and their integral curves

Definition 1. A smooth map $v\colon M \to TM$ such that

$$\pi \circ v = id_M \qquad \Longleftrightarrow \qquad v(m) \in T_mM$$

is called a (smooth) *vector field* on M.

For example, the map

$$v\colon S^1 \to \mathbb{R}^2, \qquad v(x) = \big(x,\ (-x_1, x_0)\big)$$

is a (smooth) vector field on S^1. Since the first component of v must be x by the very definition of a vector field, usually one simply writes

$$v(x) = (-x_1, x_0). \tag{2}$$

Denote

$$\mathfrak{X}(M) := \big\{ v\colon M \to TM \text{ is a vector field } \big\}.$$

Clearly, $\mathfrak{X}(M)$ is a real vector space with respect to the following operations:

- $\big(v_1 + v_2\big)(m) := v_1(m) + v_2(m)$, where $v_1, v_2 \in \mathfrak{X}(M)$;
- $\big(\lambda v\big)(m) = \lambda v(m)$, where $v \in \mathfrak{X}(M)$ and $\lambda \in \mathbb{R}$.

In fact, any vector field can be multiplied by any smooth function:

$$\big(f \cdot v\big)(m) = f(m)v(m), \qquad \text{where } v \in \mathfrak{X}(M) \text{ and } f \in C^\infty(M).$$

We summarize this in the following.

Proposition 3. *The set $\mathfrak{X}(M)$ of all vector fields on M has the structure of a module over $C^\infty(M)$ with respect to the pointwise addition and multiplication.* □

Example 4. Consider $M = \mathbb{R}^k$. We have seen that $T\mathbb{R}^k \cong \mathbb{R}^k \times \mathbb{R}^k$ and that the natural projection equals π_1. Hence, a vector field is a map of the form

$$v(x) = \big(x, y(x)\big),$$

where $y \in C^\infty(\mathbb{R}^k; \mathbb{R}^k)$. Hence, we can identify $\mathfrak{X}(\mathbb{R}^k)$ with $C^\infty(\mathbb{R}^k; \mathbb{R}^k)$ via the map

$$v = \big(id_{\mathbb{R}^k},\ y\big) \mapsto y.$$

More formally, this map is an isomorphism of $C^\infty(M)$-modules.

Generalizing the above example slightly, pick a chart (U, φ) on a manifold M. Since

$$\mathrm{v}_\varphi(m) := \big([\gamma_1^m], \dots, [\gamma_k^m]\big), \qquad \text{where} \quad \gamma_j^m(t) := \varphi^{-1}\big(\varphi(m) + te_j\big),$$

is a basis of T_mM, we can find the coordinates $\big(y_1(m), \dots, y_k(m)\big)$ of $v(m)$ with respect to this basis. In other words, $y\colon U \to \mathbb{R}^k$ is a map such that

$$v(m) = \mathrm{v}_\varphi(m) \cdot y(m)$$

holds at any point $m \in U$. Notice that the map y is well defined even if v is not necessarily smooth. This map is called *the coordinate (or local) representation of* v with respect to the chart (U, φ).

Proposition 5. *The map $v\colon M \to TM$ satisfying $\pi \circ v = id_M$ is a smooth vector field if an only if for each chart (U, φ) as above the coordinate representation y of v is smooth.*

Proof. Recall that for any chart (U, φ) on M as above we constructed a chart $\big(\pi^{-1}(U),\ \tau_\varphi^{-1}\big)$ on TM. Just by the definitions of τ_φ and y, for the coordinate representation of v with respect to these charts we have

$$\tau_\varphi^{-1} \circ v \circ \varphi^{-1} = \big(x,\ y \circ \varphi^{-1}(x)\big).$$

Hence, v is smooth if and only if y is smooth. □

Thus, locally over each chart U vector fields can be identified with smooth vector-valued maps just as in Example 4. It turns out, however, that in general no such identification can exist.

Let $\gamma\colon (a, b) \to M$ be a smooth curve. At any point $t \in (a, b)$ we define *the tangent vector* $\dot\gamma(t) \in T_{\gamma(t)}M$ to γ by

$$\dot\gamma(t) := \big[s \mapsto \gamma_t(s)\big] \qquad \text{where} \quad \gamma_t(s) := \gamma(t + s).$$

Definition 6 (Integral curves)**.** A (smooth) curve γ is called *an integral curve* of a vector field v if

$$\dot\gamma(t) = v\big(\gamma(t)\big)$$

holds for any $t \in (a, b)$.

Example 7. Consider the curve $\gamma\colon \mathbb{R} \to S^1$, $\gamma(t) = (\cos t,\ \sin t)$. We have $\dot\gamma(t) = (-\sin t,\ \cos t)$. Furthermore, if v is given by (2), then

$$v \circ \gamma\,(t) = (-\sin t,\ \cos t).$$

Hence, γ is an integral curve of (2).

Let us consider integral curves on $\mathbb{R}^k$ in some detail. Thus, represent a vector field $v \in \mathfrak{X}(\mathbb{R}^k)$ by a smooth map $y\colon \mathbb{R}^k \to \mathbb{R}^k$ just as in Example 4 above. A map $\gamma\colon (a, b) \to \mathbb{R}^k$ is an integral curve of v if and only if

$$\dot\gamma(t) = y\big(\gamma(t)\big) \qquad \iff \qquad \begin{cases} \dot\gamma_1(t) = y_1\big(\gamma_1(t), \dots, \gamma_k(t)\big), \\ \qquad \cdots\cdots\cdots \\ \dot\gamma_k(t) = y_k\big(\gamma_1(t), \dots, \gamma_k(t)\big), \end{cases} \tag{8}$$

holds for any $t \in (a, b)$. In other words, an integral curve of a vector field is a solution of a system of ordinary differential equations (ODEs). Notice that the map y does not depend on t, that is (8) is *an autonomous* system of ODEs.

Conversely, any system of ODEs as above, is uniquely specified by a map $y \in C^\infty(\mathbb{R}^k; \mathbb{R}^k)$. In view of Example 4, y corresponds to a vector field v, whose integral curves are solutions of the initial system of ODEs. Thus, at least for Euclidean spaces, integral curves of vector fields and solutions of autonomous systems of ODEs are synonymous.

Exercise 9. Show that if γ is a C^1-curve satisfying (8), then γ is smooth.

Notice that for autonomous systems we have the following property: If γ is a solution of (8) such that $\gamma(t_0) = m_0$, then for any $c \in (a, b)$

$$\gamma_c(t) := \gamma(t + c), \qquad t \in (a - c, b - c)$$

is also a solution. In other words, the integral curve γ_1 of v such that $\gamma_1(t_1) = m_0$ satisfies

$$\gamma_1(t) = \gamma\big(t + t_0 - t_1\big),$$

that is γ_1 differs from γ just by a shift of time. For this reason, one often chooses $t_0 = 0$ as the initial time for integral curves of vector fields.

By the main theorem of ODEs [Hal80, Sec.I.3], we obtain the following existence and uniqueness result.

Theorem 10. *Let v be a smooth vector field on an open subset $\Omega \subset \mathbb{R}^k$. For any point $m_0 \in \Omega$ there exists a neighbourhood $V \subset \Omega$ of m_0 and a number $\varepsilon > 0$ with the following property: For any $m \in V$ there exists an integral curve*

$$\gamma = \gamma_m\colon (-\varepsilon, \varepsilon) \to \Omega \qquad \textit{such that} \qquad \gamma(0) = m.$$

This integral curve is unique in the following sense: If $\beta\colon (-\delta, \delta) \to M$ is any other integral curve such that $\beta(0) = m$, then β and γ_m coincide on $(-\varepsilon, \varepsilon) \cap (-\delta, \delta)$. Moreover, the map

$$\Phi\colon (-\varepsilon, \varepsilon) \times V \to \mathbb{R}^k, \qquad \Phi(t, m) := \gamma_m(t) \tag{11}$$

is smooth. □

Definition 12. An integral curve $\gamma\colon (a, b) \to M$ of a vector field v is called *maximal*, if the following property holds: For any other integral curve $\beta\colon (c, d) \to M$ of v such that for some $t_0 \in (a, b) \cap (c, d)$ we have $\gamma(t_0) = \beta(t_0)$, then:

(i) $(c, d) \subset (a, b)$;

(ii) $\beta = \gamma\big|_{(c,d)}$.

It is a well-known fact from the theory of ODEs, that for any $m_0 \in \mathbb{R}^k$ there is a unique maximal solution of (8) through m_0. A straightforward corollary is, that for any vector field v on any manifold M there is a unique maximal integral curve γ of v through a given point.

Corollary 13. *If M is compact, then a maximal integral curve of any vector field is defined on all of $\mathbb{R}$.*

Proof. For each point $m \in M$ pick a chart (U, φ) containing m. Hence, we obtain the coordinate representation of the vector field v via the map $y\colon \Omega := \varphi(U) \to \mathbb{R}^k$. Then $\gamma\colon (a, b) \to U$ is an integral curve of v if and only if for $\Gamma := \varphi \circ \gamma$ we have

$$\dot\Gamma(t) = y\big(\Gamma(t)\big) \qquad \text{for } t \in (a, b),$$

cf. (8). By Theorem 10, there exists a neighborhood V_m such that for each $\hat m \in V_m$ the integral curve $\gamma_{\hat m}$ through $\hat m$ is defined on $(-\varepsilon_m, \varepsilon_m)$. By the compactness of M, we can find a finite collection of points $\{m_1, \ldots, m_\ell\}$ such that the corresponding collection of neighbourhoods $\big\{V_j := V_{m_j} \mid 1 \le j \le \ell\big\}$ covers all of M. Set

$$\varepsilon := \frac{\min\{\varepsilon_{m_j} \mid 1 \le j \le \ell\}}{2} > 0.$$

Let $\gamma\colon (a,b)\to M$ be a maximal integral curve of v. Assuming $b<\infty$, the point $m_0 := \gamma(b-\varepsilon)$ lies in some V_j. By the construction of ε, there is a unique integral curve γ_{m_0}, which is well-defined on $(-2\varepsilon, 2\varepsilon)$ and satisfies $\gamma_{m_0}(0)=m_0$. Set

$$\hat\gamma\colon (a,\, b+\varepsilon)\to M,\qquad \hat\gamma(t)=\begin{cases}\gamma(t) & \text{for } t\in(a,\, b-\varepsilon),\\ \gamma_{m_0}\big(t-b+\varepsilon\big) & \text{for } t\in\big[b-\varepsilon,\, b+\varepsilon\big).\end{cases}$$

Notice that $\hat\gamma$ is continuous since $\gamma_{m_0}(b-\varepsilon)=m_0=\gamma(b-\varepsilon)$. In fact, by construction $\hat\gamma$ is an integral curve of v on $(a,b-\varepsilon)\cup(b-\varepsilon,b+\varepsilon)$. It follows that $\hat\gamma$ is a C^1-integral curve of v and therefore smooth by Exercise 9. Thus, $\hat\gamma$ is an integral curve of v defined on a larger interval. This contradicts the maximality of γ. □

0.2 Flows and 1-parameter groups of diffeomorphisms

In this section I assume that M is a compact manifold.

For a vector field v define *the flow* of v to be the map

$$\Phi\colon \mathbb{R}\times M\to M,\qquad \Phi(t,m)=\gamma_m(t).$$

Of course, this is just the map Φ of Theorem 10 extended to the whole real line. Sometimes, (11) is referred to as *the local flow* of v.

Beside the flow, for each fixed $t\in\mathbb{R}$ it is also convenient to consider

$$\Phi_t\colon M\to M,\qquad \Phi_t(m)=\Phi(t,m)=\gamma_m(t).$$

Proposition 14. *The following holds:*

(i) Each Φ_t is a diffeomorphism. Moreover, $\Phi_t^{-1}=\Phi_{-t}$;

(ii) For any $t,s\in\mathbb{R}$ we have $\Phi_t\circ\Phi_s=\Phi_{t+s}=\Phi_s\circ\Phi_t$;

(iii) $\Phi_0=id_M$;

Proof. For $m\in M$ and $t\in\mathbb{R}$ denote $\Phi_t(m)=\hat m$. This means that $\gamma_m(t)=\hat m$, where γ_m is an integral curve of v such that $\gamma_m(0)=m$.

Consider the curve β defined by $\beta(s)=\gamma_m(s+t)$. Then β is an integral curve of v and $\beta(0)=\gamma_m(t)=\hat m$, that is $\beta=\gamma_{\hat m}$. Hence,

$$\Phi_s(\hat m)=\gamma_{\hat m}(s)=\beta(s)=\gamma_m(s+t)=\Phi_{s+t}(m)\qquad\Longleftrightarrow\qquad \Phi_s\circ\Phi_t=\Phi_{s+t}.$$

Since *(iii)* holds by the very definition of Φ_t, by *(ii)* we obtain

$$\Phi_{-t}\circ\Phi_t=id_M=\Phi_t\circ\Phi_{-t}.$$

In particular, each Φ_t is a diffeomorphism and $\Phi_t^{-1}=\Phi_{-t}$ □

Definition 15. A 1-parameter group of diffeomorphisms is any smooth map $\Phi\colon\mathbb{R}\times M\to M$ such that Properties *(i)*–*(iii)* of Proposition 14 hold.

To explain the above definition, notice that the set

$$\mathrm{Diff}(M):=\big\{f\colon M\to M \mid f \text{ is a diffeomorphism}\,\big\}$$

is a group with respect to the composition operation. $\mathrm{Diff}(M)$ is called *the diffeomorphism group* of M. With this understood, a 1-parameter group of diffeomorphisms is simply a homomorphism of groups

$$\mathbb{R} \to \mathrm{Diff}(M), \qquad t \mapsto \Phi_t$$

such that $\Phi_t(m) = \Phi(t, m)$ depends smoothly on (t, m).

Thus, Proposition 14 states that each vector field on a compact manifold generates a 1-parameter group of diffeomorphisms. Conversely, it turns out that any 1-parameter group of diffeomorphisms generates a vector field in the following sense.

Proposition 16. *For any 1-parameter group of diffeomorphisms Φ there exists a vector field v, whose 1-parameter group of diffeomorphisms coincides with Φ.*

Proof. For any $m \in M$ denote

$$\gamma_m \colon \mathbb{R} \to M, \qquad \gamma_m(t) := \Phi(t, m) \qquad \text{and} \qquad v(m) := \dot\gamma_m(0).$$

The reader should check that v is a smooth vector field.

Furthermore, denote $\gamma_m(t) = \hat m$ and observe that

$$\gamma_{\hat m}(s) = \Phi_s(\hat m) = \Phi_s\big(\Phi_t(m)\big) = \Phi_{t+s}(m) = \gamma_m(t+s). \tag{17}$$

In other words, if $a_t \colon \mathbb{R} \to \mathbb{R}$ is defined by $a_t(s) = t + s$, then $\gamma_{\hat m} = \gamma_m \circ a_t$. Hence,

$$v\big(\gamma_m(t)\big) = v\big(\hat m\big) = \dot\gamma_{\hat m}(0) = [\gamma_{\hat m}(s)]_{s=0} = [\gamma_m(t+s)]_{s=0} = \dot\gamma_m(t),$$

where the first three equalities follow straight from corresponding definitions and the fours one follows from (17). Thus, γ_m is the integral curve of v. Therefore, the 1-parameter group of diffeomorphisms generated by v is

$$(t, m) \mapsto \gamma_m(t) = \Phi(t, m),$$

In other words, the 1-parameter group of diffeomorphisms generated by v coincides with Φ. □

To sum up, for compact manifolds there is a natural bijective correspondence between vector fields and 1-parameter groups of diffeomorphisms.

Bibliography

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