Assignment 10

MATH-F310: Differential Geometry I

December 12, 2022

1. $SU(2) := U(2) \cap SL(2, \mathbb{C})$, i.e., unitary 2×2 matrices with determinant one. Now show that the map $f : S^3 \subset \mathbb{C}^2 \to SU(2)$,
$$f(\alpha, \beta) = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}$$
is a homeomorphism ($\alpha, \beta \in \mathbb{C}$). And hence we can construct charts on $SU(2)$ to prove that it's a smooth manifold of dimension 3.

2. Define a function $f : S^2 \to \mathbb{R}$ by $f(x, y, z) = x^2 + 2y + z + 2$. Compute the differential of f at the point $(0, 1, 0)$. Try to compute all the critical points of f.

3. **Smooth Urysohn Theorem:** If A and B are disjoint, closed subsets of a smooth manifold X, Prove that there is a smooth function f on X, such that $0 \leq f \leq 1$ with $f = 0$ on A and $f = 1$ on B. [Hint: partition of unity.]

4. **Morse function on a manifold:** Suppose that a point $x \in \mathbb{R}^k$, is a nondegenerate critical point of a function $f : \mathbb{R}^k \to \mathbb{R}$. We define the matrix
$$(h_{ij}) = \big(\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\big)$$
to be the Hessian of f at x. If the Hessian is non-singular at the critical point a, one says a is a *nondegerate* critical point of f. The concept of nongeneracy makes sense on manifolds, via local parametrizations. Suppose that $f : X \to \mathbb{R}$ has a critical point at $x \in X$ and that ϕ is a local parametrization carrying the origin to x. Then 0 is a critical point for the function $f \circ \phi$, for $d_0(f \circ \phi) = d_x f \circ d\phi_0$. We shall declare x to be *nondegerate* for f if 0 is nondegenerate for $f \circ \phi$. The difficulty with such local definitions is that one must always prove the cloice of parametrization to be unimportant. In this case, if ϕ_1 and ϕ_2 are two choices, then $f \circ \phi_1 = (f \circ \phi_2) \circ \psi$, where $\psi = \phi_2^{-1} \circ \phi_1$. Now prove that:

   a. Suppose that f is a function on $\mathbb{R}^k$ with a nondegenerate critical point at 0, and ψ is a diffeomorphism with $\psi(0) = 0$. Then $f \circ \psi$ is also a nondegenerate critical point at 0. Observe that this result makes the nondegenerate points on a manifold well-defined. A function is *Morse* if all the critical points are nondegerate.

   b. Suppose that $f = \sum_{i,j} a_{ij} x_i x_j$ in $\mathbb{R}^k$. Check that its Hessian matrix is $H = (a_{ij})$. Considering $\mathbb{R}^k$ as the vector space of column vectors, H operates as a linear map by left multiplication, as usual. Show that if $Hv = 0$, then f is critical all along the line through v and 0. Thus the origin is an isolated critical point iff H is non-singular.

   c. Show that the height function $h : (x_1, x_2, x_3, x_4) \mapsto x_4$ on the sphere S^3 is a Morse function with two critical points, the poles. Note that one pole is a maximum and the other a minimum.