Assignment 1

MATH-F310: Differential Geometry I

September 29, 2022

1. Let $a, b, c \in \mathbb{R}$ be such that $ac - b^2 > 0$. Show that the maximum and minimum values of the function $g(x_1, x_2) = x_1^2 + x_2^2$ on the ellipsoid $ax_1^2 + 2bx_1x_2 + cx_2^2 = 1$ are of the form $\frac{1}{\lambda_1}$ and $\frac{1}{\lambda_2}$ where λ_1 and λ_2 are eigenvalues of the matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$.

2. For what values of c, $f^{-1}(c)$ is a surface in $\mathbb{R}^3$?
   $(a) f(x, y, z) = x^2 + y^2 + z^2$
   $(b) f(x, y, z) = xyz$

3. Let C be the circle of radius r in the YZ-plane centerted at the point $(0, a, 0)$, where $a > r$. And we obtain T by rotating C around the X-axis. More formally

   $$T := \{(\sqrt{x^2 + y^2} - a)^2 + z^2 = r^2\}$$

   Show that T is indeed a surface.

4. Show that $\{(x, y, z) \in \mathbb{R}^3 | y^2 = x^3\}$ is not a surface in $\mathbb{R}^3$.

5. Show that the maximum and the minimum values of the function $g(x_1, \ldots, x_{n+1}) = \sum_{i,j=1}^{n+1} a_{ij}x_ix_j$ on the unit n-sphere $x_1^2 + \cdots + x_{n+1}^2 = 1$, where (a_{ij}) is a symmetric $n \times n$ matrix of real numbers, are eigenvalues of the matrix (a_{ij}).