Assignment 6

MATH-F310: Differential Geometry I

November 4, 2022

1. Suppose that a surface S is a union $S = S_1 \cup S_2$ where S_1 and S_2 are two orientable surfaces such that $S_1 \cap S_2$ is connected. Prove that S is also orientable.

2. **Möbius band:** Define a map $f : \mathbb{R}^2 \to \mathbb{R}^3$ by

$$f(u, v) = \big((2 - v \sin \frac{u}{2}) \sin u, (2 - v \sin \frac{u}{2}) \cos u, v \cos \frac{u}{2}\big)$$

   Show that the image of f (Möbius band) is a surface in $\mathbb{R}^3$.

3. Show that any tangent plane to $z = x^2 - y^2$ intersects the surface in two perpendicular lines.

4. Let $f : S_1 \to S_2$ be a local diffeomorphism between two surfaces S_1 and S_2 with S_2 orientable. Let N_2 be a unit normal field on S_2. We define a map $N_1 : S_1 \to \mathbb{R}^3$ as follows: if $p \in S_1$, we put

$$N_1(p) = \frac{a \times b}{|a \times b|}$$

   where a, b form a basis of $T_p S_1$ satisfying

$$\det\big((Df_p)(a), (Df_p)(b), N_2(f(p))\big) > 0$$

   Show that N_1 is a unit normal field on S_1 and consequently S_1 is also orientable.

5. Show that if a surface S in $\mathbb{R}^3$ is represented both as $f^{-1}(c)$ and $g^{-1}(d)$ where $\nabla f(p) \neq 0, \nabla g(p) \neq 0 \forall p \in S$, then $\forall p \in S, \nabla f(p) = \lambda(p) \nabla g(p)$ for some $\lambda(p) \neq 0$.