Assignment 8

MATH-F310: Differential Geometry I

December 1, 2022

1. Let V be a real-vector space of dimension n. Prove that V is a smooth manifold of dimension n and the map $T_xV \to V, [\gamma] \mapsto \gamma'(0)$ gives us a natural isomorphism between T_xV and V $\forall x \in V$. γ being a smooth curve in V with $\gamma(0) = x$.

2. Show that $S^2 \times S^2$ is a smooth manifold.

3. Consider the function
$$f : \mathbb{RP}^2 \to \mathbb{R}, \;\; f([x : y : z]) = \frac{xy + yz + xz}{x^2 + y^2 + z^2}$$
Show that f is smooth and find all critical points of f. Notice that a point $m := [x_0, y_0, z_0]$ is called critical for f, if for any smooth curve $\gamma : (-\epsilon, \epsilon) \to \mathbb{RP}^2$ such that $\gamma(0) = m$, we have $d_m f([\gamma]) := \frac{d}{dt}\big|_{t=0}(f \circ \gamma(t)) = 0$.

4. (a) Let $M(2, \mathbb{R})$ be the set of 2×2 matrices and $SL(2, \mathbb{R}) := \{A \in M(2, \mathbb{R}) \,|\, \det(A) = 1\}$. Prove that $SL(2, \mathbb{R})$ is a smooth 3-manifold.

   (b) Notice that the set of trace free 2×2 matrices form a vector space of dimension 3 inside $M(2, \mathbb{R})$. We call it $\mathfrak{sl}(2, \mathbb{R})$. Prove that the 3 matrices $E_1 := \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, E_2 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, E_3 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ form a basis of $\mathfrak{sl}(2, \mathbb{R})$. Now find three smooth curves γ_1, γ_2 and γ_3 such that $\gamma_i(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{Id}$ and $\gamma'_i(0) = E_i$ for $i = 1, 2, 3$. Hence we proved that $T_{\mathrm{Id}}SL(2, \mathbb{R}) = \mathfrak{sl}(2, \mathbb{R})$. We call $\mathfrak{sl}(2, \mathbb{R})$ the Lie algebra of $SL(2, \mathbb{R})$.

5. Let $\varphi : \mathbb{R}^{k+1} \to \mathbb{R}$ be a smooth function such that $\nabla\varphi(x) \neq 0 \, \forall x \in \varphi^{-1}(0) =: M$. Show that M is a smooth manifold of dimension n, and the tangent space at a point $m \in M, T_mM$ is naturally isomorphic to $\nabla\varphi(m)^{\perp}$.