# Differential Geometry I

## Lecture notes

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This is a draft. If you spot a mistake, please e-mail me: andriy(DOT)haydys@ulb.be.

# Contents

# Chapter 1

# Introduction

A substantial part of mathematics is related to solving equations of various types. Given any equation, we may try to analyze this by studying the following sequence of questions:

1. Does there exist a solution (a root)?
2. If the answer to the previous question is affirmative, how many solutions does the equation have?
3. If there are finitely many solutions, can we find all of them?

For example, the reader learned at school the properties of the quadratic equation $ax^2+bx+c=0$. In this case the above questions are easy to settle and the answers are well known to the reader.

Sometimes an equation may have an infinite number of solutions. If there are only countably many roots, the last question from the list above still makes sense. For example, all solutions of the equation $\sin x = 0$ are given by a simple formula: $x_n = \pi n,\ n \in \mathbb{Z}$.

In many cases, however, equations have uncountably many solutions so that asking to find all solutions is not really meaningful. Instead, it turns out to be more interesting to replace Question 3 by the following one:

3′. What are the properties of the set of *all solutions*?

Which particular properties we are interested in may depend on the context. The property most relevant to the content of this course is concerned with the local structure of the set of all solutions.

Let us consider an example. The equation

$$x_1^2 + x_2^2 + x_3^2 = 1, \tag{1.1}$$

where $x_1, x_2, x_3 \in \mathbb{R}$, clearly has uncountably many solutions.

Denote $S^2 := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$, that is $S^2$ is the set of all solutions of (1.1). Of course, $S^2$ is the sphere of radius 1, however let us pretend for a moment that we do not know this. As a subset of $\mathbb{R}^3$, $S^2$ is a topological space. It turns out that this topological space has a very particular property, which we consider in some detail next.

The familiar stereographic projection from the north pole $N := (0, 0, 1)$ is given by

$$\varphi_N\colon S^2 \setminus \{N\} \to \mathbb{R}^2, \qquad \varphi_N(x) = \Big(\frac{x_1}{1-x_3},\ \frac{x_2}{1-x_3}\Big).$$

This is in fact a homeomorphism with the inverse

$$\varphi_N^{-1}(y) = \frac{1}{1+y_1^2+y_2^2}\big(2y_1,\ 2y_2,\ -1+y_1^2+y_2^2\big), \qquad y = (y_1, y_2) \in \mathbb{R}^2. \tag{1.2}$$

We can also define a stereographic projection from the south pole $S := (0,0,-1)$ by

$$\varphi_S\colon S^2 \setminus \{S\} \to \mathbb{R}^2, \qquad \varphi_S(x) = \Big(\frac{x_1}{1+x_3},\ \frac{x_2}{1+x_3}\Big),$$

which is also a homeomorphism.

Since any point on the sphere lies either in $S^2 \setminus \{N\}$ or $S^2 \setminus \{S\}$ (or both), any point on the sphere has a neighbourhood, which is homeomorphic to an open subset of $\mathbb{R}^n$ (of course, $n = 2$ in our particular example and the open subset is $\mathbb{R}^2$ itself). This property leads to the notion of *a manifold*, which will play a cenral rôle in the course. We will see below, that this property is *not* specific to Equation (1.1). On the contrary, for any smooth map $F\colon \mathbb{R}^k \to \mathbb{R}^\ell$ and almost any $c \in \mathbb{R}^\ell$ the set of all solutions to the equation $F(x) = c$ is a manifold. That is, there is a huge pull of examples of manifolds and many objects of particular interest in mathematics turn out to be manifolds.

Coming back to our example, we compute:

$$\varphi_S \circ \varphi_N^{-1}(y) = \Big(\frac{y_1}{|y|^2},\ \frac{y_2}{|y|^2}\Big). \tag{1.3}$$

Hence, $\varphi_S \circ \varphi_N^{-1}$ is *smooth* on an open subset $\mathbb{R}^2 \setminus \{0\}$ and a similar computation yields that this is also true for $\varphi_N \circ \varphi_S^{-1}$. This property can be used to study smooth functions on the sphere directly without reference to the ambient space. More importantly, in more general situations where the ambient Euclidean space may be simply absent, an analogue of this property allows one to apply familiar tools of analysis to functions defined on more sophisticated objects than just subsets of an Euclidean space. In some sense, this constitutes the core of differential geometry.

Summing up, the aim of these notes is to transfer familiar tools of mathematical analysis to a more geometric setting where the underlying domain of a function (map) is not just an open subset of $\mathbb{R}^n$, but rather a manifold. The benefits of doing so are ubiquitous, but explaining this in some detail requires a bit of work. It is my hope to convey that the notion of a manifold is useful and well worth studying further.

# Chapter 2

# Smooth manifolds

## 2.1 Basic definitions and examples

Recall that a topological space $M$ is called Hausdorff, if for any two distinct points $m_1, m_2 \in M$ there are neighbourhoods $U_1 \ni m_1$ and $U_2 \ni m_2$ such that $U_1 \cap U_2 = \varnothing$. If the topology of $M$ admits a countable base, then $M$ is said to be second countable. For example, $\mathbb{R}^k$ is both Hausdorff and second countable.

**Definition 2.1.** A Hausdorff second countable topological space $M$ is called *a topological manifold* of dimension $k$, if $M$ is locally homeomorphic to $\mathbb{R}^k$.

To explain, this means that any point $m \in M$ admits a neighbourhood $U$ and a homeomorphism $\varphi\colon U \to V$, where $V$ is an open subset of $\mathbb{R}^k$. The pair $(U, \varphi)$ (or, sometimes just $U$) is called a chart on $M$ near $m$.

Notice that the requirements that a manifold is Hausdorff and second countable are to a great extent of technical nature, whereas being locally homeomorphic to $\mathbb{R}^k$ is a crucial property of manifolds.

Clearly, $\mathbb{R}^k$ and in fact any open subset of $\mathbb{R}^k$ are examples of topological manifolds of dimension $k$. As we have established in the introduction, 2-spheres are manifolds of dimension two. Similar arguments yield in fact that the $k$-sphere

$$S^k := \Big\{(x_1, \dots, x_{k+1}) \in \mathbb{R}^{k+1} \mid \sum_{j=1}^{k+1} x_j^2 = 1\Big\}$$

is a $k$-manifold.

Somewhat special is the case of dimension zero. Since $\mathbb{R}^0$ is by definition a single point, the above definition requires that each point of $M$ has a neighborhood consisting only of this point. In other words, $M$ is a countable discrete space.

**Definition 2.2.** A collection of charts $\mathcal{U} = \{(U_\alpha, \phi_\alpha) \mid \alpha \in A\}$ is called a $C^0$-*atlas*, if $\bigcup_{\alpha\in A} U_\alpha = M$, that is if any point of $M$ is contained in some chart. Here $A$ is an arbitrary index set.

For example, $\mathbb{R}^k$ admits a $C^0$-atlas consisting of a single chart $(\mathbb{R}^k, id)$. In the introduction we have constructed a $C^0$-atlas on the 2-sphere consisting of two charts. However, there is no $C^0$-atlas on $S^2$ consisting of a single chart, since $S^2$ is not homeomorphic to an open subset of $\mathbb{R}^2$ (why?).

Given a $C^0$-atlas $\mathcal{U}$, pick any two charts $(U_\alpha, \varphi_\alpha)$ and $(U_\beta, \varphi_\beta)$ such that $U_\alpha \cap U_\beta \neq \varnothing$. The map

$$\theta_{\alpha\beta} := \varphi_\alpha \circ \varphi_\beta^{-1} \colon \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta), \tag{2.3}$$

which a homeomorphism between two open subsets of $\mathbb{R}^k$, is called *a coordinate transformation*[1]. Notice that $\theta_{\beta\alpha}$ is the inverse map to $\theta_{\alpha\beta}$. In particular, $\theta_{\alpha\beta}$ is a homeomorphism between $\varphi_\beta(U_\alpha \cap U_\beta)$ and $\varphi_\alpha(U_\alpha \cap U_\beta)$.

It is a common practice to suppress the domain and the target of $\theta_{\alpha\beta}$ writing simply $\theta_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1}$. While this may be confusing at first, the advantage is that this allows us to suppress less important details so that the most essential features are clearer. If in doubt, the reader should write the domain and target explicitly.

**Definition 2.4.** A $C^0$-atlas $\mathcal{U}$ is called *smooth*, if all coordinate transformation maps $\theta_{\alpha\beta}$, $\alpha, \beta \in A$, are smooth.

*Remark* 2.5. Equally well, we can say that $\mathcal{U}$ is a $C^\ell$-atlas, if all coordinate transformation maps belong to $C^\ell(\mathbb{R}^n; \mathbb{R}^n)$ (keep in mind that these are defined on open subsets of $\mathbb{R}^n$ only) for some fixed natural number $\ell$. The theory does not depend much on the choice of $\ell$ as long as $\ell$ is not too small. In practice $\ell \geq 3$ would suffice in most of the cases, however to avoid non-essential details it is convenient to put $\ell = \infty$ from the very beginning.

Two charts $(U, \varphi)$ and any $(V, \psi)$ not necessarily from the same atlas are said to be *smoothly compatible* if the maps

$$\varphi \circ \psi^{-1} \qquad \text{and} \qquad \psi \circ \varphi^{-1} \tag{2.6}$$

are smooth, compare with (2.3). We consider two atlases $\mathcal{U}$ and $\mathcal{V}$ as "essentially equal", if all charts from $\mathcal{U}$ are smoothly compatible with all charts in $\mathcal{V}$. More formally, we have the following definition.

**Definition 2.7.** Two atlases $\mathcal{U}$ and $\mathcal{V}$ on the same underlying topological space $M$ are called equivalent, if $\mathcal{U} \cup \mathcal{V}$ is a smooth atlas on $M$, that is if all charts from $\mathcal{U}$ are smoothly compatible with all charts in $\mathcal{V}$. An equivalence class of atlases is called *a smooth structure* on $M$. *A smooth manifold* consists of a Hausdorff second countable topological space and a smooth structure.

To explain the point of the above definition, consider the 2-sphere. In the introduction we constructed a smooth atlas on $S^2$, namely $\mathcal{U} := \big\{ (S^2 \setminus \{N\},\ \varphi_N),\ (S^2 \setminus \{S\},\ \varphi_S) \big\}$. However, there are many ways to construct another smooth atlas, for example as follows:

$$\mathcal{U}' := \big\{ S^2 \setminus \{N\}, \varphi_N \big\} \cup \big\{ (S^2_+, \varphi_+) \big\}.$$

Here $S^2_+ := \big\{ x \in S^2 \mid x_3 > 0 \big\}$ and $\varphi_+(x) = (x_1, x_2)$.

**Exercise 2.8.** Check that $\mathcal{U}'$ is a smooth atlas equivalent to $\mathcal{U}$.

It should be intuitively clear, that the description of $S^2$ via smooth atlases $\mathcal{U}$ and $\mathcal{U}'$ are 'essentially equal'. Hence, it is natural to identify $(S^2, \mathcal{U})$ and $(S^2, \mathcal{U}')$.

An atlas $\mathcal{U}$ is called *maximal*, if for any chart $(V, \psi)$ smoothly compatible with all charts in $\mathcal{U}$ is already contained in $\mathcal{U}$.

The importance of maximal atlases stems from the following result.

**Lemma 2.9.** *Each equivalence class of smooth atlases is represented by a unique maximal atlas.*

[1] The origin of this terminology will be clear below.

*Proof.* For a smooth atlas $\mathcal{U}$ on $M$ define

$$\mathcal{U}_{max} := \big\{(V,\psi) \text{ is a chart on } M \text{ s.t. (2.6) are both smooth for all } (U,\varphi)\in\mathcal{U}\big\}.$$

**Exercise 2.10.** Check that $\mathcal{U}_{max}$ is a smooth atlas on $M$.

By the construction of $\mathcal{U}_{max}$, we have $\mathcal{U}\subset\mathcal{U}_{max}$. Hence, any chart smoothly compatible with any chart in $\mathcal{U}_{max}$ is also smoothly compatible with any chart in $\mathcal{U}$ and therefore is contained in $\mathcal{U}_{max}$. Hence, $\mathcal{U}_{max}$ is maximal. Clearly, $\mathcal{U}$ and $\mathcal{U}_{max}$ represent the same smooth structure. □

By the above lemma, a smooth manifold may be considered as being equipped with a maximal atlas. In particular, if $\mathcal{U}$ is any smooth atlas on $M$, we may freely add any chart smoothly compatible with all charts in $\mathcal{U}$ without changing the smooth structure. For example, if $(U,\varphi)$ is a chart near $m_0$, then $(U,\hat\varphi)$ with

$$\hat\varphi(m)=\varphi(m)-\varphi(m_0)$$

is also a chart near $m_0\in M$ smoothly compatible with all charts in $\mathcal{U}$. The chart $(U,\hat\varphi)$ satisfies

$$\hat\varphi(m_0)=0,$$

which is commonly expressed by saying that $(U,\hat\varphi)$ is *centered at* $m_0$.

*Remark* 2.11. In what follows only smooth manifolds will be considered. Therefore, by saying that $M$ is a manifold, we always mean a *smooth* manifold, unless explicitly stated otherwise.

Let us finish this section with some further examples of manifolds.

*Example* 2.12 (Products). Let $M$ and $N$ be smooth manifolds of dimensions $k$ and $\ell$ respectively. Let $\mathcal{U}=\big\{(U_\alpha,\varphi_\alpha)\mid\alpha\in A\big\}$ and $\mathcal{V}=\big\{(V_\lambda,\psi_\lambda)\mid\lambda\in\Lambda\big\}$ be smooth atlases on $M$ and $N$ respectively. Then the product $M\times N$ is a Hausdorff second countable topological space. We define a $C^0$-atlas on $M\times N$ by setting

$$\mathcal{W}:=\Big\{\big(U_\alpha\times V_\lambda,\ \varphi_\alpha\times\psi_\lambda\big)\mid\alpha\in A,\ \lambda\in\Lambda\Big\}.$$

Given any two charts $\big(U_\alpha\times V_\lambda,\ \varphi_\alpha\times\psi_\lambda\big)$ and $\big(U_\beta\times V_\mu,\ \varphi_\beta\times\psi_\mu\big)$ the corresponding coordinate transformation is given by $\theta_{\alpha\beta}\times\eta_{\lambda\mu}$, where $\theta_{\alpha\beta}=\varphi_\alpha\circ\varphi_\beta^{-1}$ and $\eta_{\lambda\mu}=\psi_\lambda\circ\psi_\mu^{-1}$ are smooth maps. More precisely, this means the following:

$$\begin{aligned}\theta_{\alpha\beta}\times\eta_{\lambda\mu}&\colon\mathbb{R}^k\times\mathbb{R}^\ell\to\mathbb{R}^k\times\mathbb{R}^\ell,\\ \theta_{\alpha\beta}\times\eta_{\lambda\mu}(x,y)&=\big(\theta_{\alpha\beta}(x),\ \eta_{\lambda\mu}(y)\big),\quad x\in\mathbb{R}^k,\ y\in\mathbb{R}^\ell.\end{aligned}$$

In particular, $\theta_{\alpha\beta}\times\eta_{\lambda\mu}$ is a smooth map, which means that the atlas constructed above is smooth. Hence, $M\times N$ is a smooth manifold of dimension $k+\ell$. This yields in particular that the following

(i) the $k$-dimensional torus $\mathbb{T}^k:=S^1\times\cdots\times S^1$ and

(ii) the cylinder $\mathbb{R}\times S^1$

are smooth manifolds. In the latter case, the dimension of $\mathbb{R}\times S^1$ equals 2.

*Example* 2.13 (Real projective spaces). The real projective space $\mathbb{RP}^k$ of dimension $k$ is defined to be the set of all lines in $\mathbb{R}^{k+1}$ through the origin. Since each line through the origin is uniquely determined by a point on this line distinct from the origin, we have

$$\mathbb{RP}^k = \big(\mathbb{R}^{k+1} \setminus \{0\}\big)/\sim,$$

where $x, y \in \mathbb{R}^{k+1} \setminus \{0\}$ are defined to be equivalent if and only if there exists $\lambda \in \mathbb{R} \setminus \{0\}$ such that $y = \lambda x$. In particular, we have the canonical surjective quotient map

$$\pi\colon \mathbb{R}^{k+1} \setminus \{0\} \to \mathbb{RP}^k, \qquad \pi(x) = [x].$$

If $x = (x_0, x_1, \dots, x_k) \in \mathbb{R}^{k+1} \setminus \{0\}$, it is customary to write $[x_0 : x_1 : \ldots : x_k]$ for $[x]$.

We endow $\mathbb{RP}^k$ with the quotient topology, that is $U \subset \mathbb{RP}^k$ is open if and only $\pi^{-1}(U)$ is open in $\mathbb{R}^{k+1} \setminus \{0\}$. It is straighforward to check that this yields a Hausdorff second countable topological space.

To construct a $C^0$-atlas on $\mathbb{RP}^k$, observe that each

$$U_j := \big\{[x_0 : x_1 : \ldots : x_k] \in \mathbb{RP}^k \mid x_j \neq 0\big\}, \qquad j = 0, 1, \dots, k,$$

is an open subset of $\mathbb{RP}^k$. Indeed, this follows from the fact that

$$\pi^{-1}(U_j) = \big\{(x_0, \dots, x_k) \in \mathbb{R}^{k+1} \setminus \{0\} \mid x_j \neq 0\big\}$$

is an open subset of $\mathbb{R}^{k+1} \setminus \{0\}$.

The map

$$\begin{aligned} &\varphi_j\colon U_j \to \mathbb{R}^k, \\ &\varphi_j\big[x_0 : x_1 : \ldots : x_{j-1} : x_j : x_{j+1} : \ldots : x_k\big] = \Big(\frac{x_0}{x_j}, \frac{x_1}{x_j}, \dots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \dots, \frac{x_k}{x_j}\Big) \end{aligned}$$

is well-defined and continuous. Moreover, the map

$$\psi_j\colon \mathbb{R}^k \to U_j, \qquad \psi_j\big(y_0, y_1, \dots, y_{k-1}\big) = \big[y_0 \colon y_1 \colon \ldots \colon y_{j-1} \colon 1 \colon y_j \colon \ldots \colon y_{k-1}\big] \tag{2.14}$$

is a continuous inverse of $\varphi_j$, that is $\varphi_j$ is a homeomorphism. Since the collection $U_0, \dots, U_k$ clearly covers all of $\mathbb{RP}^k$, $\mathcal{U} := \big\{(U_j, \varphi_j) \mid j = 0, 1, \dots, k\big\}$ is a $C^0$-atlas on $\mathbb{RP}^k$.

Next, let us consider the coordinate transformations. To simplify the notations we consider only the map $\theta_{01} = \varphi_0 \circ \varphi_1^{-1} = \varphi_0 \circ \psi_1$. We have

$$\theta_{01}(y_0, \dots, y_{k-1}) = \varphi_0\big([y_0 : 1 : y_1, \dots, y_{k-1}]\big) = \Big(\frac{1}{y_0}, \frac{y_1}{y_0}, \dots, \frac{y_{k-1}}{y_0}\Big),$$

which is smooth on

$$\varphi_1\big(U_0 \cap U_1\big) = \big\{y \in \mathbb{R}^k \mid y_0 \neq 0\big\}.$$

A similar argument yields that all coordinate transformations $\theta_{ij} = \varphi_i \circ \psi_j$ are smooth on their domains of definition. Thus, $\mathcal{U}$ is a smooth atlas and $\mathbb{RP}^k$ is a smooth manifold of dimension $k$.

It may be useful to keep some non-examples of manifolds in mind.

(a) The set $M = \{(x, y) \in \mathbb{R}^2 \mid x^2 = y^2\}$ consisting of two straight lines $y = \pm x$ intersecting at the origin, is not a manifold. Indeed, if $M$ were a manifold, its dimension must be one. However, the origin does not have a neighbourhood in $M$ homeomorphic to an open subset of $\mathbb{R}^1$ (Why?).

(b) A disjoint union of manifolds is a manifold. However, a disjoint uncountable union of non-empty manifolds is *not* a manifold, since the second countability axiom is violated. For example, the disjoint union of real lines labeled by $\alpha \in (0,1)$

$$N := \bigsqcup_{\alpha\in(0,1)} \mathbb{R}_\alpha$$

is not a manifold. Notice that the above example is *not* homeomorphic to $(0,1)\times\mathbb{R}$, which is a manifold indeed, since, for example, each line $\mathbb{R}_\alpha \subset N$ is an open subset.

(c) Consider the following line "with a double point":

$$L := (-\infty, 0) \cup \{a, b\} \cup (0, +\infty).$$

Here $\{a, b\}$ is understood as a set consisting of two distinct elements. The following two subsets

$$U_a := (-\infty, 0) \cup \{a\} \cup (0, +\infty) \qquad \text{and} \qquad U_b := (-\infty, 0) \cup \{b\} \cup (0, +\infty)$$

cover all of $L$. Define $\varphi_a\colon U_a \to \mathbb{R}$ by $\varphi_a(x) = x$ if $x \neq a$ and $\varphi_a(a) = 0$. By the same token we can define $\varphi_b\colon U_b \to \mathbb{R}$.

A topology on $L$ is defined simply by saying that $V$ is open if and only if $\varphi_a(V\cap U_a)$ and $\varphi_b(V\cap U_b)$ are open in $\mathbb{R}$.

This yields a second countable topological space with a smooth atlas. However, $L$ is non-Hausdorff.

L 1

## 2.2 Smooth maps

Given a smooth structure on $M$, we can make sense of smoothness of functions defined on $M$ as follows.

**Definition 2.15.** Let $M$ be a manifold with a smooth structure represented by a smooth atlas $\mathcal{U} = \{(U_\alpha, \varphi_a)\}$. A function $f\colon M \to \mathbb{R}$ is said to be smooth, if for any chart $(U_\alpha, \varphi_a)$ the function $f\circ\varphi_\alpha^{-1}\colon \mathbb{R}^k \to \mathbb{R}$ is smooth.

Notice that since an open subset $V$ of $M$ is again a smooth manifold, it makes sense to say that a function is smooth on $V$. The smoothness of functions is then a local property in the following sense: $f$ is smooth if and only if the restriction of $f$ to any open subset of $M$ is smooth. In particular, if $\{V_\alpha \mid \alpha \in A\}$ is an open covering of $M$ and $f$ is smooth on each $V_\alpha$, then $f$ is smooth on $M$.

Strictly speaking, we still have to show that the notion of smoothness in Definition 2.15 is independent of the choice of an atlas. Indeed, assume that $f$ is smooth with respect to $\mathcal{U}$ and pick an atlas $\mathcal{V} = \{(V_\mu, \psi_\mu)\}$ equivalent to $\mathcal{U}$. Then on $\psi_\mu(U_\alpha\cap V_\mu) \subset \mathbb{R}^n$ we have

$$f\circ\psi_\mu^{-1}\big|_{\psi_\mu(U_\alpha\cap V_\mu)} = f\circ\varphi_\alpha^{-1}\circ\varphi_\alpha\circ\psi_\mu^{-1}\big|_{\psi_\mu(U_\alpha\cap V_\mu)} = f\circ\varphi_\alpha^{-1}\circ\theta_{\alpha\mu}\big|_{\psi_\mu(U_\alpha\cap V_\mu)},$$

where $\theta_{\alpha\mu} = \varphi_\alpha\circ\psi_\mu^{-1}$ is a smooth map. Hence, $f$ is smooth with respect to $\mathcal{V}$ on any subset $U_\alpha\cap V_\mu$. Since these subsets cover all of $M$, $f$ is smooth on $M$ with respect to $\mathcal{V}$.

*Example* 2.16. Let $F\colon \mathbb{R}^3 \to \mathbb{R}$ be any smooth function. Define $f\colon S^2 \to \mathbb{R}$ as the restriction of $F$ to $S^2$. I claim that $f$ is a smooth function on $S^2$. Indeed, let $(U, \varphi)$ be any chart on $S^2$ constructed in the introduction. For concreteness, let us pick the chart $\big(S^2 \setminus \{N\}, \varphi_N\big)$. Then

$$f \circ \varphi_N^{-1}(y_1, y_2) = F\Big(\frac{2y_1}{1+y_1^2+y_2^2}, \frac{2y_2}{1+y_1^2+y_2^2}, \frac{-1+y_1^2+y_2^2}{1+y_1^2+y_2^2}\Big), \qquad (y_1, y_2) \in \mathbb{R}^2.$$

Hence, $f \circ \varphi^{-1}$ is smooth and it is clear that this is also the case for $\big(S^2 \setminus \{S\}, \varphi_S\big)$. Thus, $f$ is a smooth function on $S^2$.

*Example* 2.17. Let $F\colon \mathbb{R}^{k+1} \setminus \{0\} \to \mathbb{R}$ be a smooth homogeneous function of degree $0$, that is $F(\lambda x) = F(x)$ for all $\lambda \in \mathbb{R} \setminus \{0\}$ and $x \in \mathbb{R}^{k+1} \setminus \{0\}$. Define $f\colon \mathbb{RP}^k \to \mathbb{R}$ by setting $f\big([x]\big) = F(x)$. This yields a well-defined function, which I claim is smooth. Indeed, pick any chart $(U_j, \varphi_j)$ constructed in Example 2.13. Using (2.14), we obtain

$$f \circ \varphi_j^{-1}(y_0, \dots, y_{k-1}) = F(y_0, \dots, y_{j-1}, 1, y_j, \dots, y_{k-1}),$$

which is smooth everywhere on $\mathbb{R}^k$. Hence, $f$ is smooth.

**Proposition 2.18.** *The set $C^\infty(M)$ of all smooth functions on a manifold $M$ is an algebra, that is*

- *$f, g \in C^\infty(M)$, $\lambda, \mu \in \mathbb{R} \implies \lambda f + \mu g \in C^\infty(M)$;*
- *$f, g \in C^\infty(M) \implies f \cdot g \in C^\infty(M)$.*

*Proof.* Let $f, g$ be any two smooth functions and $\lambda, \mu$ two real numbers. For any chart $(U, \varphi)$ the functions

$$(\lambda f + \mu g) \circ \varphi^{-1} = \lambda\big(f \circ \varphi^{-1}\big) + \mu\big(g \circ \varphi^{-1}\big),$$
$$(f \cdot g) \circ \varphi^{-1} = f \circ \varphi^{-1} \cdot g \circ \varphi^{-1}$$

are clearly smooth, hence $\lambda f + \mu g$ and $f \cdot g$ are smooth functions on $M$. □

Let $f\colon M \to \mathbb{R}^\ell$ be a map, which can be written as an $\ell$-tuple of functions: $f = (f_1, \dots, f_\ell)$. We say that $f$ is smooth, if each component $f_j$ is a smooth function on $M$.

It is also possible to define the notion of smoothness for maps between manifolds. To this end, let $M$ and $N$ be two manifolds of dimensions $k$ and $\ell$ respectively. Pick an atlas $\mathcal{U}$ on $M$ and an atlas $\mathcal{V}$ on $N$.

**Definition 2.19.** A continuous map $f\colon M \to N$ is said to be smooth, if for any $(U, \varphi) \in \mathcal{U}$ and any $(V, \psi) \in \mathcal{V}$ the map

$$\psi \circ f \circ \varphi^{-1}\colon \mathbb{R}^k \to \mathbb{R}^\ell$$

is smooth.

*Remark* 2.20. The requirement that $f$ is continuous in the above definition is only needed to ensure that $\psi \circ f \circ \varphi^{-1}$ is defined on an open subset of $\mathbb{R}^k$. The map $\psi \circ f \circ \varphi^{-1}$ is called *the coordinate presentation of $f$* (with respect to charts $(U, \varphi)$ and $(V, \psi)$).

The argument used to verify that the notion of smoothness of a function is well-defined is very common in the theory of manifolds and will be typically omitted below. However, the reader may wish to prove the following proposition as an exercise.

**Proposition 2.21.** *If $f\colon M \to N$ and $g\colon N \to L$ are smooth maps between smooth manifolds, then $g \circ f$ is also smooth.* □

**Definition 2.22.** A smooth map $f\colon M \to N$ such that $f^{-1}\colon N \to M$ exists and is also smooth is called *a diffeomorphism*.

Observe that if $(U,\varphi)$ is a chart, then $\varphi\colon U \to \varphi(U) \subset \mathbb{R}^k$ is a diffeomorphism.

To obtain a somewhat non-trivial example of a diffeomorphism, consider the tangent function:

$$\tan\colon \big(-\pi/2,\ \pi/2\,\big) \to \mathbb{R}, \qquad \tan(x) = \frac{\sin x}{\cos x}.$$

This is smooth, bijective, the inverse function $\arctan$ exists and is smooth. Hence the interval $(-\pi/2,\ \pi/2)$ is diffeomorphic to $\mathbb{R}$. In fact any open interval is diffeomorphic to $\mathbb{R}$ (Why?).

A standard non-example is given by the map

$$f\colon \mathbb{R} \to \mathbb{R}, \qquad f(x) = x^3,$$

which is clearly smooth and bijective. The inverse map however fails to be smooth at the origin so that $f$ is *not* a diffeomorphism.

If there exists a diffeomorphism between $M$ and $N$, we say that $M$ and $N$ are diffeomorphic. Notice that in this case we must have $k = \dim M = \dim N = \ell$. Indeed, if $f$ is a diffeomorphism between $M$ and $N$, then $F := \psi \circ f \circ \varphi^{-1}$ is a diffeomorphism between open subsets of $\mathbb{R}^k$ and $\mathbb{R}^\ell$. Let us denote $G := F^{-1} = \varphi \circ f^{-1} \circ \psi^{-1}$, which is smooth by the definition of smoothness for $f^{-1}$. Diferentiating $G \circ F = \mathrm{id}_{\mathbb{R}^k}$ at the point $G(x)$, we obtain

$$id_{\mathbb{R}^k} = D_{G(x)}\big(id_{\mathbb{R}^k}\big) = D_{G(x)}\big(G \circ F\big) = D_{F(G(x))}G \cdot D_{G(x)}F = D_x G \cdot D_{G(x)}F,$$

where $id_{\mathbb{R}^k}$ is the identity map. In particular, $DG$ is surjective at each point. Furthermore, by a similar argument applied to the identity $F \circ G = \mathrm{id}_{\mathbb{R}^\ell}$, we obtain that $DG$ is injective at each point. In other words, $D_y G\colon \mathbb{R}^\ell \to \mathbb{R}^k$ is a linear isomorphism, which is only possible if $k = \ell$.

**Definition 2.23.** A map $f\colon M \to N$ is called *a local diffeomorphism*, if for any point $m \in M$ there exists an open neighbourhood $U \ni m$ in $M$ and an open neighbourhood $V \ni f(m)$ in $N$ such that

$$f\big|_U\colon U \to V$$

is a diffeomorphism.

A non-trivial example of a local diffeomorphism can be obtained as follows. The map

$$f\colon \mathbb{R} \to S^1, \qquad f(x) = \big(\sin x,\ \cos x\big)$$

is a local diffeomorphism (why?), which is not a diffeomorphism, since $f(0) = f(\pm 2\pi) = f(\pm 4\pi) = \dots$

A non-trivial result from the course of analysis we need here is the following.

**Theorem 2.24.** *Let $U$ be open in $\mathbb{R}^n$ and $f\colon U \to \mathbb{R}^k$ be smooth. Assume that at some $x \in U$ the differential $D_x f$ of $f$ is invertible. Then $n = k$ and $f$ is a local diffeomorphism at $x$, that is there exist open subsets $U' \ni x$ and $V \ni f(x)$ such that*

$$f\big|_{U'}\colon U' \to V$$

*is a diffeomorphism.*

A proof of this theorem can be found for example in [BT03, Thm 9.4.1].

## 2.3 The fundamental theorem of algebra

As an application of the notions introduced in the preceding sections, we prove the fundamental theorem of algebra in this section. This requires some additional notions and constructions, which are of independent interest.

Let $M$ and $N$ be two manifolds of dimensions $k$ and $\ell$ respectively. Pick a smooth map $f\colon M \to N$, a point $m \in M$ and charts $(U_0, \varphi_0)$ and $(V_0, \psi_0)$ such that $m \in U$ and $f(m) \in V$.

**Definition 2.25.** We say that $m$ is *a critical point of $f$* if the differential of the coordinate representation

$$D_{\varphi_0(m)}\big(\psi_0 \circ f \circ \varphi_0^{-1}\big) \tag{2.26}$$

is non-surjective at $\varphi_0(m)$.

**Lemma 2.27.** *The notion of a critical point is well-defined, i.e., this is independent of the choice of charts.*

*Proof.* Pick any charts $(U_1, \varphi_1)$ and $(V_1, \psi_1)$ such that $m \in U_1$ and $f(m) \in V_1$. We have

$$\psi_0 \circ f \circ \varphi_0^{-1} = \psi_0 \circ \psi_1^{-1} \circ \psi_1 \circ f \circ \varphi_1^{-1} \circ \varphi_1 \circ \varphi_0^{-1} = \theta_{01}^{\psi} \circ \big(\psi_1 \circ f \circ \varphi_1^{-1}\big) \circ \theta_{10}^{\varphi},$$

which is valid on an open subset containing $m$. Hence, by the chain rule, we obtain for the differentials

$$D\big(\psi_0 \circ f \circ \varphi_0^{-1}\big) = D\theta_{01}^{\psi} \circ D\big(\psi_1 \circ f \circ \varphi_1^{-1}\big) \circ D\theta_{10}^{\varphi}.$$

Since $D\theta_{01}^{\psi}$ and $D\theta_{10}^{\varphi}$ are invertible everywhere on the domain of their definition, we obtain that $D\big(\psi_0 \circ f \circ \varphi_0^{-1}\big)$ is non-surjective at $\varphi_0(m)$ if and only if $D\big(\psi_1 \circ f \circ \varphi_1^{-1}\big)$ is non-surjective at $\varphi_1(m)$. This finishes the proof of this lemma. □

**Definition 2.28.** Any non-critical point is called *regular*. We say that $n \in N$ is *a regular value* of $f$, if $f^{-1}(n)$ consists of regular points only. If $f^{-1}(n)$ contains at least one singular point, then $n$ is called *a singular value* of $f$.

Let me stress that any point, which does not lie in the image of $f$, is a regular value of $f$ (this fact of course follows from the definition but may be easily missed at first).

Notice that in the particular case $k = \ell$, (2.26) is a linear map $\mathbb{R}^k \to \mathbb{R}^k$. Hence, (2.26) is non-surjective if and only if it has a non-trivial kernel, or, still if and only if $\det D_{\varphi_0(m)}\big(\psi_0 \circ f \circ \varphi_0^{-1}\big) = 0$.

With these preliminaries at hand, we can prove the following.

**Theorem 2.29** (The fundamental theorem of algebra)**.** *Let $p(z) := a_k z^k + a_{k-1}z^{k-1} + \dots + a_1 z + a_0$ be a polynomial with complex coefficients of degree $k \geq 1$. Then $p$ has at least one (complex) root.*

*Proof.* Identify $\mathbb{R}^2$ with $\mathbb{C}$ by writing

$$y = (y_1, y_2) \equiv y_1 + y_2 i = z.$$

For a fixed polynomial $p = a_k z^k + \dots + a_1 z + a_0$ such that $a_k \neq 0$, where $k \geq 1$ , define a map $f\colon S^2 \to S^2$ by the rule

$$f(x) = \begin{cases} N, & \text{if } x = N, \\ \varphi_N^{-1} \circ p \circ \varphi_N(x), & \text{if } x \neq N. \end{cases} \tag{2.30}$$

The proof proceeds in a number of steps.

**Step 1.** *$f$ is smooth.*

It is enough to check that $f$ is smooth near $N$, i.e., that $\varphi_S \circ f \circ \varphi_S^{-1}$ is smooth. To see this, consider

$$\varphi_S \circ f \circ \varphi_S^{-1} = \varphi_S \circ \varphi_N^{-1} \circ \varphi_N \circ f \circ \varphi_N^{-1} \circ \varphi_N \circ \varphi_S^{-1} = \theta_{SN} \circ p \circ \theta_{SN}^{-1},$$

which is valid on $\mathbb{C} \setminus \{0\}$. By (1.3),

$$\theta_{SN}(z) = \frac{z}{z\bar{z}} = \frac{1}{\bar{z}}\,.$$

Consequently, for $z \neq 0$ we have

$$\varphi_S \circ f \circ \varphi_S^{-1}(z) = \frac{1}{\overline{p(1/\bar{z})}} = \frac{1}{\frac{\bar{a}_k}{z^k} + \cdots + \frac{\bar{a}_1}{z} + \bar{a}_0} = \frac{z^k}{\bar{a}_k + \cdots + \bar{a}_1 z^{k-1} + \bar{a}_0 z^k}.$$

Since $\varphi_S \circ f \circ \varphi_S^{-1}(0) = 0$, $\varphi_S \circ f \circ \varphi_S^{-1}$ is clearly smooth on $\mathbb{C}$. Hence, $f$ is smooth as claimed.

**Step 2.** *The differential of the map $z \mapsto p(z)$ at the point $z$ can be identified with $h \mapsto p'(z)h$, where*

$$p'(z) = ka_k z^{k-1} + \ldots 2a_2 z + a_1.$$

Denote $p(z) = u(y_1, y_2) + v(y_1, y_2)i$. Since $p$ is a holomorphic function of $z$, by the Cauchy-Riemann equations we have

$$\begin{aligned} D_z p(h) &= \begin{pmatrix} \frac{\partial u}{\partial y_1} & \frac{\partial u}{\partial y_2} \\ \frac{\partial v}{\partial y_1} & \frac{\partial v}{\partial y_2} \end{pmatrix} \begin{pmatrix} h_1 \\ \\ h_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial y_1} h_1 - \frac{\partial v}{\partial y_1} h_2 \\ \frac{\partial v}{\partial y_1} h_1 + \frac{\partial u}{\partial y_1} h_2 \end{pmatrix} \\ &\equiv \Big( \frac{\partial u}{\partial y_1} h_1 - \frac{\partial v}{\partial y_1} h_2 \Big) + \Big( \frac{\partial v}{\partial y_1} h_1 + \frac{\partial u}{\partial y_1} h_2 \Big) i \\ &= \frac{\partial p}{\partial y_1}\, h = p'(z)\, h. \end{aligned}$$

**Step 3.** *The set of critical values of $f$ is finite.*

Indeed, any critical point of $f$ is either $N$ or of the form $\varphi_N^{-1}(z)$, where $z$ is a zero of the polynomial

$$p'(z) = ka_k z^{k-1} + \ldots 2a_2 z + a_1.$$

Hence, the number of critical points of $f$ is finite and therefore the number of critical values is also finite.

*Remark* 2.31. Notice that $N$ is a critical point of $f$ as long as $k \geq 2$, which we can assume without loss of generality, since any polynomial of degree $1$ obviously has a root.

**Step 4.** *For any regular value $x \in S^2$, the number of points in $f^{-1}(x)$ is finite and independent of $x$.*

First notice that the set $\mathcal{R}(f)$ of regular values is open and connected as a complement of a finite number of points in $S^2$.

Furthermore, for any $n \in \mathcal{R}(f)$ and any $m \in f^{-1}(n)$, the map $\varphi_N \circ f \circ \varphi_N^{-1} = p$ is a local diffeomorphism at $\varphi(m)$ by Theorem 2.24. Since $\varphi_N$ is a homeomorphism, $f$ is a local homeomorphism at $m$. In particular, there is a neighbourhood $U' \ni m$ such that $U' \cap f^{-1}(n) =$

$\{m\}$. Thus, $f^{-1}(n)$ is discreet. Since $S^2$ is compact, $f^{-1}(n)$ is also compact as a closed subset of a compact space. Hence, $f^{-1}(n)$ must in fact be a finite set.

The above argument actually shows that the map

$$\mathcal{R}(f) \ni x \mapsto \# f^{-1}(x) \in \mathbb{N} \tag{2.32}$$

is locally constant. Since $\mathcal{R}(f)$ is connected, this function must be constant.

**Step 5.** *We prove this theorem.*

Observe first that (2.32) cannot vanish everywhere on $\mathcal{R}(f)$. Indeed, the image of $f$ is obviously infinite, whereas the set of critical values is finite by Step 3. Hence, there are regular values, which are in the image of $f$.

In fact, (2.32) vanishes nowhere as a locally constant function on an open connected space. Hence, $f^{-1}(S) \neq \varnothing$ as long as $S \in \mathcal{R}(f)$. Also, if $S$ is a critical value, then $f^{-1}(S)$ contains at least one critical point. In either case, $f^{-1}(S)$ is non-empty, which means that $p$ has at least one root. $\square$

The above proof turns out to contain a few ideas which can be used in other circumstances too. However, this requires some technical results, which are proved first. L 2

## 2.4 Tangent spaces

We begin with the following consideration. Let $\gamma$ be a smooth curve in $\mathbb{R}^k$ through some $p \in \mathbb{R}^k$, that is a smooth map $\gamma\colon (a,b) \to \mathbb{R}^k$, such that $\gamma(t_0) = p$ for some $t_0 \in (a,b)$. Recall that the tangent vector of $\gamma$ at $p$ is

$$\dot\gamma(t_0) := \frac{d}{dt}\Big|_{t=t_0} \gamma(t) \in \mathbb{R}^k.$$

Let now $\gamma$ be a smooth curve on the 2-sphere through some $p \in S^2$. Since $S^2$ is a subset of $\mathbb{R}^3$, we may think of $\gamma$ as a curve in $\mathbb{R}^3$ satisfying

$$\gamma_1^2(t) + \gamma_2^2(t) + \gamma_3^2(t) = 1 \qquad \forall t \in (a,b). \tag{2.33}$$

It is reasonable to call the set

$$T_pS^2 := \big\{ \mathrm{v} \in \mathbb{R}^3 \mid \mathrm{v} \text{ is the tangent vector of some smooth curve on } S^2 \text{ through } p \big\}$$

the tangent space of $S^2$ at the point $p$.

To determine $T_pS^2$ more explicitly, differentiate (2.33) with respect to $t$ and set $t = t_0$:

$$\gamma_1(t_0)\dot\gamma_1(t_0) + \gamma_2(t_0)\dot\gamma_2(t_0) + \gamma_3(t_0)\dot\gamma_3(t_0) = 0 \qquad \Longleftrightarrow \qquad \langle p, \dot\gamma(t_0)\rangle = 0.$$

In other words, the tangent vector of any smooth curve on $S^2$ through $p$ is necessarily orthogonal to $p$. Moreover, it is clear that any vector orthogonal to $p$ arises in this way (consider all great circles through $p$). Hence,

$$T_pS^2 = p^\perp.$$

However, for an abstract manifold $M$ a smooth curve $\gamma$ on $M$ does not lie in an Euclidean space in any obvious way so that the above definition of $T_pS^2$ does not immediately generalize. A nice workaround goes as follows.

Let $\gamma$ be a smooth curve through $m \in M$. We can assume that $\gamma$ is defined on $(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$ and $\gamma(0) = m$.

**Definition 2.34.** Two smooth curves $\gamma_1$ and $\gamma_2$ through $m$ as above are said to be equivalent, if for some chart $(U, \varphi)$ such that $m \in U$ we have

$$\frac{d}{dt}\Big|_{t=0}\big(\varphi \circ \gamma_1(t)\big) = \frac{d}{dt}\Big|_{t=0}\big(\varphi \circ \gamma_2(t)\big). \tag{2.35}$$

In other words, $\gamma_1 \sim \gamma_2$ if and only if the smooth curves $\varphi \circ \gamma_1$ and $\varphi \circ \gamma_1$ in $\mathbb{R}^k$ (!) have the same tangent vector.

Furthermore, an equivalence class of smooth curves is called *a tangent vector* at $m$. The set $T_mM$ of all tangent vectors at $m$ is called *the tangent space* at $m$.

**Exercise 2.36.** Show that the above equivalence relation is independent of the choice of a chart.

**Exercise 2.37.** Show that the tangent space to $S^2$ at some $p \in S^2$ in the sense of Definition 2.34 is $p^\perp$.

While Definition 2.34 is has a clear geometric meaning, the algebraic structure of $T_mM$ is opaque in this approach. For this reason we adopt an alternative definition, which is more algebraic.

First notice that given any tangent vector $[\gamma]$ through $m$ we can define the map $\partial_{[\gamma]}\colon C^\infty(M) \to \mathbb{R}$ by setting

$$\partial_{[\gamma]}(f) = \frac{d}{dt}\Big|_{t=0} f\big(\gamma(t)\big) = \lim_{t\to 0}\frac{f\big(\gamma(t)\big) - f(m)}{t}\,.$$

**Proposition 2.38.** *The map $\partial_{[\gamma]}$ is well defined and has the following properties:*

*(i)* *$\partial_{[\gamma]}$ is $\mathbb{R}$-linear, that is*

$$\partial_{[\gamma]}\big(\lambda f + \mu g\big) = \lambda\, \partial_{[\gamma]}(f) + \mu\, \partial_{[\gamma]}(g)$$

*holds for all $\lambda, \mu \in \mathbb{R}$ and $f, g \in C^\infty(M)$.*

*(ii)* *$\partial_{[\gamma]}$ satisfies*

$$\partial_{[\gamma]}(fg) = \partial_{[\gamma]}(f)g(m) + f(m)\partial_{[\gamma]}(g)$$

*for all $f, g \in C^\infty(M)$.*

*Proof.* We only need to prove that $\partial_{[\gamma]}$ is well-defined, since *(i)* and *(ii)* are clear from the definition.

Thus, pick two smooth equivalent curves $\gamma_1$ and $\gamma_2$ through $m$. Pick also a chart $(U, \varphi)$ near $m$ and denote $F := f \circ \varphi^{-1}\colon \mathbb{R}^k \to \mathbb{R}$ and $\beta_j := \varphi \circ \gamma_j\colon (-\varepsilon, \varepsilon) \to \mathbb{R}^k$. Notice, that

$$\dot\beta_1(0) = \dot\beta_2(0) =: \mathrm{v},$$

since $\gamma_1$ and $\gamma_2$ are equivalent.

We have

$$\frac{d}{dt}\Big|_{t=0}\Big(f \circ \gamma_1(t)\Big) = \frac{d}{dt}\Big|_{t=0}\Big(f \circ \varphi^{-1} \circ \varphi \circ \gamma_1(t)\Big) = \frac{d}{dt}\Big|_{t=0}\Big(F \circ \beta_1(t)\Big) = D_{\mathrm{v}}F\big(\varphi(m)\big),$$

where $D_{\mathrm{v}}F$ is the derivative of $F$ in the direction of v, that is

$$D_{\mathrm{v}}F = \sum_{j=1}^{k}\frac{\partial F}{\partial x_j}\mathrm{v}_j = \langle \nabla F,\, \mathrm{v}\rangle. \tag{2.39}$$

A similar computation yields also

$$\frac{d}{dt}\Big|_{t=0}\Big(f\circ\gamma_2(t)\Big) = D_{\mathrm{v}}F\big(\varphi(m)\big),$$

thus demonstrating that $\partial_{[\gamma]}$ depends on the equivalence class of $\gamma$ only as the notation suggests. $\square$

Motivated by the above proposition we give the following

**Definition 2.40.** An $\mathbb{R}$-linear map $\partial\colon C^\infty(M)\to\mathbb{R}$ satisfying the Leibnitz rule

$$\partial(fg) = \partial(f)\,g(m) + f(m)\,\partial(g) \qquad \forall f,g\in C^\infty(M)$$

is called *a derivation at* $m$.

Notice that a constant function, which takes value $1$ everywhere on $M$, is annihilated by any derivation. Indeed, this follows from the following computation:

$$\partial(1) = \partial(1^2) = \partial(1)\cdot 1 + 1\cdot\partial(1) = 2\,\partial(1).$$

By the linearity of derivations, any constant function is annihilated by each derivation.

For the proof of Proposition 2.42 below, we need the following technical result, whose proof is deferred till the next section.

**Proposition 2.41.** *For any manifold $M$ and any point $m_0\in M$ the following holds.*

*(i) Suppose $f$ is a smooth function defined on a neighborhood $U$ of $m_0$. Then there is a smooth function $\hat f$ defined everywhere on $M$ and a neighborhood $\hat U\subset U$ of $m_0$ such that $f$ and $\hat f$ coincide everywhere on $\hat U$.*

*(ii) Let $\partial$ be a derivation at $m_0$. If $f$ and $\hat f$ are two smooth functions defined everywhere on $M$ such that the restrictions of $f$ and $\hat f$ to some neighborhood $\hat U$ of $m_0$ are equal, then $\partial(f)=\partial(\hat f)$.*

Thus, for any tangent vector at $m$ we constructed an explicit derivation at $m$. It turns out that this map is a bijection as we show next.

Denote temporarily by $Der_mM$ the set of all derivations at $m$.

**Proposition 2.42.** *The map*

$$T_mM\to Der_mM,\qquad [\gamma]\mapsto\partial_{[\gamma]} \tag{2.43}$$

*is a bijection.*

*Proof.* We continue to use notations of the proof of Proposition 2.38. In addition, the chart $(U,\varphi)$ is assumed to be centered at $m_0$.

**Step 1.** (2.43) *is injective.*

Assume $\partial_{[\gamma_1]}=\partial_{[\gamma_2]}$, that is $\partial_{[\gamma_1]}(f)=\partial_{[\gamma_2]}(f)$ holds for any $f\in C^\infty(M)$. This implies in turn that

$$D_{\mathrm{v}_1}F\big(\varphi(m)\big) = D_{\mathrm{v}_2}F\big(\varphi(m)\big)$$

holds for any $F\in C^\infty(\mathbb{R}^k)$, where $\mathrm{v}_j=\dot\beta_j(0)$. Substituting $F=x_j$ in the above equality, we obtain that the $j$th components of $\mathrm{v}_1$ and $\mathrm{v}_2$ are equal for any $j$, i.e., $\mathrm{v}_1=\mathrm{v}_2$, which yields in turn that $\gamma_1$ and $\gamma_2$ are equivalent. Hence, the injectivity of (2.43) follows.

**Step 2.** *For any $F \in C^\infty(\mathbb{R}^k)$ there exist smooth functions $G_1, \dots, G_k$ such that*

$$F(x) = F(0) + \sum_{j=1}^{k} x_j G_j(x). \tag{2.44}$$

This follows by the following computation:

$$F(x) - F(0) = \int_0^1 \frac{d}{dt} F(tx)\, dt = \int_0^1 \langle \nabla F(tx), x \rangle\, dt,$$

which yields (2.44) with $G_j(x) = \int_0^1 \frac{\partial F}{\partial x_j}(tx)\, dt$.

**Step 3.** (2.43) *is surjective.*

Denote the $j$th component of $\varphi$ by $x_j$ so that $\varphi = (x_1, \dots, x_k)$. Notice that each $x_j$ is a smooth function defined on $U$.

By Proposition 2.41, we can find $\hat{U} \subset U$ and a smooth function $\hat{x}_j$ defined everywhere on $U$ such that $x_j$ and $\hat{x}_j$ coincide on $\hat{U}$.

Pick any $\partial \in Der_m M$ and define

$$\mathrm{v}_j := \partial(\hat{x}_j) \in \mathbb{R}, \qquad \mathrm{v} = (\mathrm{v}_1, \dots, \mathrm{v}_k),$$

Notice that by Proposition 2.41, *(ii)*, $\mathrm{v}_j$ does not depend on the choice of $\hat{x}_j$.

Furthermore, define

$$\begin{aligned} &\beta \colon (-\varepsilon, \varepsilon) \to \mathbb{R}^k, \qquad && \beta(t) = \big(\mathrm{v}_1 t, \dots, \mathrm{v}_k t\big), \\ &\gamma \colon (-\varepsilon, \varepsilon) \to M, && \gamma := \varphi^{-1} \circ \beta. \end{aligned}$$

By the previous step, there exist some functions $G_j \colon \mathbb{R}^k \to \mathbb{R}$ such that $f \circ \varphi^{-1}(x) = \sum x_j G_j(x)$. Hence,

$$f = f(m) + \sum_{j=1}^{k} x_j g_j, \tag{2.45}$$

where we think of $x_j$ as a function on $U$ and $g_j = G_j \circ \varphi$. In particular, $x_j(m) = 0$ for all $j = 1, \dots, k$.

Applying Proposition 2.41, *(i)* again, we can find some $\hat{g}_j$ defined globally on $M$ and a neighbourhood[2] $\hat{U} \subset U$ such that

$$f\big|_{\hat{U}} = f(m) + \sum_{j=1}^{k} \hat{x}_j \hat{g}_j\big|_{\hat{U}} = f(m) + \sum_{j=1}^{k} x_j g_j\big|_{\hat{U}}.$$

By Proposition 2.41, *(ii)* we obtain

$$\begin{aligned} \partial(f) &= \partial\big(f(m)\big) + \sum_{j=1}^{k} \partial(x_j) g_j(m) + x_j(m) \partial(g_j) \\ &= 0 + \sum_{j=1}^{k} \mathrm{v}_j g_j(m) + 0 = \sum_{j=1}^{k} \mathrm{v}_j g_j(m). \end{aligned} \tag{2.46}$$

[2] Shrinking the neighbourhoods if necessary, without loss of generality we can assume that $\hat{U}$ is the same neighbourhood for all $\hat{x}_j$ and $g_j$.

Furthermore, recalling (2.39), we obtain

$$\begin{aligned}\partial_{[\gamma]}(f) &= \sum_{j=1}^{k} \mathrm{v}_j \frac{\partial F}{\partial x_j}(0) = \sum_{j=1}^{k} \mathrm{v}_j \frac{\partial}{\partial x_j}\Big|_{x=0} \Big( f(m) + \sum_{i=1}^{k} x_i G_i(x) \Big) \\ &= \sum_{j=1}^{k} \mathrm{v}_j \Big( 0 + \sum_{i=1}^{k} \delta_{ij} G_i(0) \Big) = \sum_{j=1}^{k} \mathrm{v}_j \, g_j(m).\end{aligned}$$

Comparing this with (2.46), we conclude that $\partial(f) = \partial_{[\gamma]}(f)$ holds for any $f \in C^\infty(M)$. Hence, $\partial = \partial_{[\gamma]}$, which finishes the proof of this step and the proof of this proposition too. □

We use the bijective map of Proposition 2.42 to identify $T_mM$ with $Der_mM$. Since $Der_mM$ is clearly a vector space, we obtain the structure of a vector space on $T_mM$ in this way. Also, in view of this identification, we drop the notation $Der_mM$ in favour of $T_mM$ and we will switch freely between the two interpretations of tangent vectors as classes of curves and derivations.

**Proposition 2.47.** *For any $m \in M$ the tangent space $T_mM$ is a vector space of dimension $k = \dim M$.*

*Proof.* Pick a chart $(U, \varphi)$, $\varphi = (x_1, \dots, x_k)$ centered at $m$ as in the proof of Proposition 2.38. For each $j = 1, \dots, k$ define a curve $\gamma_j$ by

$$\varphi \circ \gamma_j(t) = (0, \dots, 0, t, 0, \dots, 0),$$

where the only non-trivial component is on the $j$th place. Correspondingly, we have $k$ derivations:

$$\partial_j := \partial_{[\gamma_j]}.$$

Notice that if $F$ is the coordinate representation of $f$, we have

$$\partial_j(f) = \frac{d}{dt}\Big|_{t=0} f \circ \gamma_j\,(t) = \frac{d}{dt}\Big|_{t=0} F(0, \dots, 0, t, 0, \dots, 0) = \frac{\partial F}{\partial x_j}(0).$$

We want to show that $\partial_1, \dots, \partial_k$ is a basis of $T_mM$.

To show that $\partial_1, \dots, \partial_k$ are linearly independent, assume there are some real numbers $\lambda_1, \dots, \lambda_k$ such that $\lambda_1 \partial_1 + \dots + \lambda_k \partial_k = 0$, that is

$$\lambda_1 \partial_1(f) + \dots + \lambda_k \partial_k(f) = 0$$

holds for any $f \in C^\infty(M)$. Substituting $f = x_j$ in the above equality[3], we obtain $\lambda_j = 0$. Hence, $\partial_1, \dots, \partial_k$ are linearly independent indeed.

Let us show that any derivation $\partial$ at $m$ can be represented as a linear combination of $\partial_1, \dots, \partial_k$. By Proposition 2.42, there exists a curve $\gamma$ through $m$ such that $\partial = \partial_{[\gamma]}$. If $\beta = \varphi \circ \gamma$ is a coordinate representation of $\gamma$ and $\mathrm{v} = \dot\beta(0)$, then

$$\partial(f) = D_{\mathrm{v}} F = \sum \mathrm{v}_j \frac{\partial F}{\partial x_j}(0) = \sum \mathrm{v}_j \, \partial_j(f).$$

Hence, $\partial = \mathrm{v}_1 \partial_1 + \dots + \mathrm{v}_k \partial_k$. □

Notice that the proof of the above proposition yields in fact a basis of $T_mM$ for any choice of a chart $(U, \varphi)$ such that $m \in U$. In fact, we have shown that $\partial_1, \dots, \partial_k$ is a basis of $T_mM$, where

$$\partial_j f = \frac{\partial}{\partial x_j}\Big|_{x=\varphi(m)} \Big( f \circ \varphi^{-1}(x) \Big). \tag{2.48}$$

L 3

Also, in the particular case $M = \mathbb{R}^k$ the proof of Proposition 2.42 shows that we have a canonical isomorphism

$$T_m\mathbb{R}^k \to \mathbb{R}^k, \qquad \partial \mapsto \big(\partial(x_1), \dots, \partial(x_k)\big). \tag{2.49}$$

Indeed, denote $\mathrm{v}_j = \partial(x_j)$ and $\mathrm{v} = (\mathrm{v}_1, \dots, \mathrm{v}_k)$. Assuming $m$ is the origin and writing $F(x) = F(0) + \sum x_j G_j(x)$ as in (2.44), we obtain

$$\partial(F) = \partial\big(F(0)\big) + \sum_{j=1}^k \Big( \partial(x_j) G_j(0) + x_j(0)\partial(G_j) \Big) = \sum_{j=1}^k \mathrm{v}_j \frac{\partial F}{\partial x_j}(0) = D_{\mathrm{v}} F(0)$$

Hence, if $\mathrm{v} = 0$, then $\partial(F) = 0$ for all $F \in C^\infty(\mathbb{R}^k)$. In other words, (2.49) is injective. This map is also surjective, since the image of $\partial = D_{\mathrm{v}}$ equals $\mathrm{v}$.

This isomorphism is particularly clear, if we interpret tangent vectors as classes of curves. Indeed, if $\partial = \partial_{[\gamma]}$, then $\partial_{[\gamma]}(x_j) = \dot\gamma_j(0)$ so that (2.49) becomes

$$[\gamma] \mapsto \dot\gamma(0). \tag{2.50}$$

*Example* 2.51. Let $f \colon \mathbb{R}^{k+1} \to \mathbb{R}$ be a smooth function. Assume that $0$ is a regular value of $f$. We shall show below that

$$M := f^{-1}(0)$$

is a smooth $k$-manifold. Taking this as granted for now, we can ask the following question: Given $m \in M$, can we describe the tangent space $T_mM$ more explicitly?

Notice that we have a natural linear map

$$\imath \colon T_mM \to Der_m\mathbb{R}^{k+1} \cong \mathbb{R}^{k+1}, \qquad \imath(\partial)h = \partial\big(h\big|_M\big). \tag{2.52}$$

If a tangent vector is interpreted as a class of curves, then we have

$$\imath\big(\partial_{[\gamma]}\big) = \big(\partial_{[\gamma]}(x_1), \dots, \partial_{[\gamma]}(x_{k+1})\big) = \dot\gamma(0) \qquad \text{or} \qquad \imath\big([\gamma]\big) = \dot\gamma(0). \tag{2.53}$$

This map is injective but not surjective. Indeed, viewing $\gamma$ as a curve in $\mathbb{R}^{k+1}$, this satisfies $f\big(\gamma(t)\big) = 0$ for all $t$. By differentiating this equation with respect to $t$, we obtain

$$\big\langle \nabla f(m), \dot\gamma(0) \big\rangle = 0.$$

Hence, $\dot\gamma(0)$ lies in the orthogonal complement to $\nabla f(m)$. Since $\dim T_mM = \dim M = k = \dim \big(\nabla f(m)\big)^\perp$, we obtain

$$\imath(T_mM) = \big(\nabla f(m)\big)^\perp.$$

Typically, this is expressed simply as $T_mM = \big(\nabla f(m)\big)^\perp$.

An interested reader may find the following exercise to be instructive: Show that the image of (2.52) equals $\big(\nabla f(m)\big)^\perp$ directly, that is without using Proposition 2.42.

[3] Technically, we should first fix an "extension" $\hat x_j$ of $x_j$ as a in the proof of Proposition 2.42 above. However, this should be clear by now and we will omit this sort of argument below.

## 2.5 Cut off and bump functions

Recall that the function

$$\lambda\colon \mathbb{R} \to \mathbb{R}, \qquad \lambda(t) := \begin{cases} 0, & \text{if } t \le 0, \\ e^{-\frac{1}{t}}, & \text{if } t > 0 \end{cases}$$

is smooth everywhere on $\mathbb{R}$ including $t = 0$.

Furthermore, notice that for any $r > 0$ we have $\lambda(t) + \lambda(r - t) > 0$ for any $t \in \mathbb{R}$. Indeed, for positive $t$ the first term is positive and for negative $t$ the second one is positive. Using this we define the function

$$\hat{\chi}_r(t) := \frac{\lambda(r-t)}{\lambda(t) + \lambda(r-t)},$$

where $r > 0$ is a parameter. Notice that $\hat{\chi}_r$ is smooth everywhere on $\mathbb{R}$, takes values in $[0, 1]$, $\chi_r(t) = 0$ for $t \ge r$, and $\chi_r(t) = 1$ for all $t \le 0$. It is convenient to define

$$\chi_r(t) := \hat{\chi}_r(t-1) = \frac{\lambda(r+1-t)}{\lambda(t-1) + \lambda(r+1-t)}, \tag{2.54}$$

which is called *a cut off function*. The graph of $\chi_r$ is shown schematically on Figure 2.1 below.

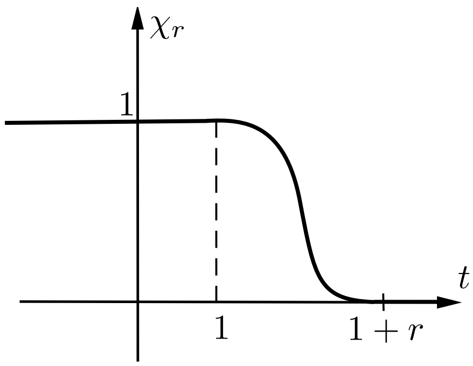


Figure 2.1: Graph of $\chi_r$.

**Proposition 2.55.** *For any point $m_0$ on $M$ and any neighbourhood $U \ni m_0$ there exists a neighbourhood $V \subset U$ and a smooth function $\rho\colon M \to [0, 1]$ such that*

$$\rho\big|_V \equiv 1 \qquad \textit{and} \qquad \rho\big|_{M \setminus U} \equiv 0.$$

*Proof.* Pick a chart centered at $m_0$. Without loss of generality, we can assume that the local homeomorphism $\varphi$ is defined everywhere on $U$. We can find $R > 0$ such that the ball $B_{2R}(0) := \{x \in \mathbb{R}^k \mid |x| < 2R\}$ is contained in $\varphi(U)$.

Furthermore, the function

$$\hat{\rho}(x) = \chi_1\left(\frac{|x|}{R}\right)$$

is smooth, equals $1$ on $B_R(0)$ and vanishes outside of $B_{2R}(0)$. Hence,

$$\rho(m) = \begin{cases} \hat{\rho}\big(\varphi(m)\big), & \text{if } m \in U, \\ 0, & \text{otherwise,} \end{cases}$$

is a well-defined smooth function, which equals $1$ on $\varphi^{-1}\big(B_R(0)\big)$ and vanishes outside of $\varphi^{-1}\big(B_{2R}(0)\big)$. $\square$

The function $\rho$ provided by the above proposition is called *a bump function*.

With these preliminaries at hand, we are ready to prove Proposition 2.41.

***Proof of Proposition 2.41.*** Let $V$ and $\rho$ be as in Proposition 2.55. The function

$$\hat f(m) = \begin{cases} f(m)\cdot\rho(m) & \text{if } m\in U\\ 0 & \text{otherwise,}\end{cases}$$

has the required properties. This proves *(i)*.

Let us prove *(ii)*. We can assume that $V\subset\hat U$ and, moreover, that $\rho$ vanishes outside of $\hat U$. Then the function $(\hat f - f)\cdot\rho$ vanishes everywhere and therefore for any derivation $\partial$ at $m_0$ we have

$$0 = \partial\big((\hat f - f)\cdot\rho\big) = \partial(\hat f - f)\cdot\rho(m_0) + \big(\hat f(m_0) - f(m_0)\big)\partial(\rho) = \partial(\hat f) - \partial(f).$$

This proves *(ii)*. $\square$

**Exercise 2.56.** Show that the open ball

$$B := \big\{x\in\mathbb{R}^k \mid |x| < 1\big\}$$

in $\mathbb{R}^k$ is diffeomorphic to $\mathbb{R}^k$. Deduce from this that any point on a manifold admits a chart $(U,\varphi)$ such that $\varphi\colon U\to\mathbb{R}^k$ is a diffeomorphism. (*Hint:* Show first that there is a diffeomorphism $f\colon(0,1)\to(0,\infty)$ such that $f(r)=r$ for all $r\le r_0<1$.)

## 2.6 The differential of a smooth map

Let $f\colon M^k\to N^\ell$ be a smooth map between two smooth manifolds.

**Definition 2.57.** For any $m\in M$ the map

$$f_*(m)\colon T_mM\to T_{f(m)}N \qquad \text{defined by} \qquad f_*(p)[\gamma] = [f\circ\gamma]$$

is called *the differential* of $f$ at the point $m$. Here $\gamma$ is a smooth curve through $m$.

Think of a tangent vector at $m\in M$ as a derivation $\partial$ at $m$. By Proposition 2.42, there exists a smooth curve $\gamma$ through $m$ such that $\partial = \partial_{[\gamma]}$. Then we have $\partial_{f_*(m)[\gamma]} = \partial_{[f\circ\gamma]}$. Hence, for any $h\in C^\infty(N)$ we obtain

$$\partial_{f_*(m)[\gamma]}(h) = \partial_{[f\circ\gamma]}(h) = \frac{d}{dt}\Big|_{t=0}\big(h\circ(f\circ\gamma)\big)(t) = \frac{d}{dt}\Big|_{t=0}\big((h\circ f)\circ\gamma\big)(t) = \partial_{[\gamma]}\big(h\circ f\big).$$

Hence, thinking of tangent vectors at $m$ as derivations at $m$ we can identify the differential of $f$ at $m$ with the map

$$\partial\mapsto f_*\partial, \qquad \text{where} \qquad \big(f_*\partial\big)h = \partial\big(h\circ f\big). \tag{2.58}$$

More precisely, this means that the following diagram commutes:

$$\begin{tikzcd} T_mM \arrow[r, "f_*(m)"] \arrow[d] & T_{f(m)}N \arrow[d] \\ Der_mM \arrow[r, "f_*(m)"] & Der_{f(m)}N. \end{tikzcd}$$

Here the vertical arrows are given by (2.43), the upper horizontal arrow represents the differential of $f$ in the sense of Definition 2.57, whereas the lower arrow represents (**??**).

Since (2.58) is obviously a linear map, we obtain the following.

**Proposition 2.59.** *The differential is a linear map.* □

Pick a chart $(U,\varphi)$ on $M$ centered at $m$ and a chart $(V,\psi)$ on $N$ centered at $f(m)\in V$. Write $\varphi=(x_1,\dots,x_k)$ and $\psi=(y_1,\dots,y_\ell)$, where $x_i$ are functions on $U$ and $y_j$ are functions on $V$ just like in the proof of Equation 2.43. By the proof of Proposition 2.47, we obtain the following bases

$$\left(\partial_1^x,\dots,\partial_k^x\right)=\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_k}\right)\qquad\text{and}\qquad\left(\partial_1^y,\dots,\partial_\ell^y\right)=\left(\frac{\partial}{\partial y_1},\dots,\frac{\partial}{\partial y_\ell}\right).\tag{2.60}$$

It is worthwhile to recall that $\partial_i^x g=\frac{\partial G}{\partial x_i}(0)$, where $G=g\circ\varphi^{-1}$ is the coordinate representation of $g$.

Since the differential is a linear map, this can be represented by a matrix relative to the above bases. Thus, a natural question arises: Can we compute the matrix of the differential relative to Bases (2.60)?

To answer this question, recall that the coordinate representation of $f$ is

$$F=\psi\circ f\circ\varphi^{-1}\colon\mathbb{R}^k\to\mathbb{R}^\ell.\tag{2.61}$$

Pick any smooth function $h$ on $N$ and consider its coordinate representation $H:=h\circ\psi^{-1}$. Then the coordinate representation of the function $g:=h\circ f$ is

$$G=g\circ\varphi^{-1}=h\circ f\circ\varphi^{-1}=h\circ\psi^{-1}\circ\psi\circ f\circ\varphi^{-1}=H\circ F.$$

In other words,

$$G(x)=H\big(F_1(x),\dots,F_\ell(x)\big),$$

where $F=\big(F_1,\dots,F_\ell\big)$. Hence, we compute:

$$\partial_i^x g=\frac{\partial G}{\partial x_i}=\sum_{j=1}^{\ell}\frac{\partial H}{\partial y_j}\,\frac{\partial F_j}{\partial x_i}=\sum_{j=1}^{\ell}\frac{\partial F_j}{\partial x_i}\,\partial_j^y h,$$

where all partial derivatives are computed at the origin, however we suppressed this in the notations to keep those simple. In other words

$$f_*(m)\,\partial_i^x=\sum_{j=1}^{\ell}\frac{\partial F_j}{\partial x_i}\,\partial_j^y\qquad\Longleftrightarrow\qquad f_*(m)\partial^x=\partial^y\cdot DF,$$

where in the last equation $\partial^x=(\partial_1^x,\dots,\partial_k^x)$ is interpreted as a $k$-tuple of vectors in $T_mM$ and, similarly, $\partial^y=(\partial_1^y,\cdots\partial_\ell^y)$; Moreover,

$$DF=\begin{pmatrix}\frac{\partial F_1}{\partial x_1} & \dots & \frac{\partial F_1}{\partial x_k}\\ \\ \frac{\partial F_\ell}{\partial x_1} & \dots & \frac{\partial F_\ell}{\partial x_k}\end{pmatrix}$$

is the Jacobi matrix of $F$ (evaluated at the origin). Thus, our computation shows that the following result holds.

**Proposition 2.62.** *Let $f\colon M^k \to N^\ell$ be a smooth map between two smooth manifolds. Pick charts $\big(U, \varphi = (x_1, \dots, x_k)\big)$ and $\big(V, \psi = (y_1, \dots, y_\ell)\big)$ centered at $m$ and $f(m)$ respectively. Denote by $F$ the coordinate representation of $f$, i.e., $F = \psi \circ f \circ \varphi^{-1}$. Then the matrix of $f_*(m)$ with respect to Bases (2.60) is given by the Jacobi matrix of $F$.* □

We finish this section by the following result.

**Proposition 2.63.** *For any two smooth maps $f\colon M \to N$ and $g\colon N \to K$ between smooth manifolds, we have*

$$(g \circ f)_*(m) = g_*\big(f(m)\big) \circ f_*(m).$$

*Proof.* The proof follows directly from the definition of the differential. Indeed, for any smooth curve $\gamma$ through $m$, we have

$$(g \circ f)_*[\gamma] = [(g \circ f) \circ \gamma] = [g \circ (f \circ \gamma)] = g_*\big(f(m)\big)\,[f \circ \gamma] = g_*\big(f(m)\big)\,\big(f_*(m)[\gamma]\big).$$

This immediately implies the statement of this proposition. □

**Exercise 2.64.** Let $A\colon \mathbb{R}^k \to \mathbb{R}^k$ be a linear map. Show that the differential of $A$ at any point can be identified with $A$ itself. More precisely, this means that the diagram

$$\begin{tikzcd} \mathbb{R}^k \arrow[r, "A_*(x)"] \arrow[d, "\Psi"'] & \mathbb{R}^k \arrow[d, "\Psi"] \\ \mathbb{R}^k \arrow[r, "A"] & \mathbb{R}^k \end{tikzcd}$$

commutes for any $x \in \mathbb{R}^k$, where $\Psi$ denotes the canonical isomorphism $T_x\mathbb{R}^k \to \mathbb{R}^k$ given by (2.49), or, equivalently, by (2.50).

L 4

# Chapter 3

# Submanifolds and partitions of unity

## 3.1 Submanifolds

Represent $\mathbb{R}^{k+\ell}$ as a product:

$$\mathbb{R}^{k+\ell} = \mathbb{R}^k \times \mathbb{R}^\ell.$$

Corresponding to this representation, we have the following maps:

$$\begin{aligned} \imath_2 &\colon \mathbb{R}^\ell \to \mathbb{R}^{k+\ell}, & \imath_2(y) &= (0, y), \\ \pi_2 &\colon \mathbb{R}^{k+\ell} \to \mathbb{R}^\ell, & \pi_2(x, y) &= y, \end{aligned}$$

where $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^\ell$.

Let $f\colon \mathbb{R}^{k+\ell} \to \mathbb{R}^\ell$ be a smooth map, which is defined on some neighbourhood $U$ of the origin. For any point $p_0 = (x_0, y_0) \in U$ we have the linear map

$$D_y f(p_0)\colon \mathbb{R}^\ell \xrightarrow{\imath_2} \mathbb{R}^{k+\ell} \xrightarrow{D_{p_0} f} \mathbb{R}^\ell. \tag{3.1}$$

For example, if $k = \ell = 1$, we have $D_y f(p_0) = \frac{\partial f}{\partial y}(p_0)$. For this reason, we call (3.1) *the partial derivative* of $f$ with respect to $y$ (at the point $p_0$).

To simplify the notations it is convenient to assume that $p_0$ is the origin and $f(0) = 0$, although this is immaterial.

**Theorem 3.2.** *If $D_y f(0)$ is an isomorphism, then there exists a smooth map $\theta\colon \mathbb{R}^{k+\ell} \to \mathbb{R}^{k+\ell}$, which is a local diffeomorpism at $0$, such that $\theta(0) = 0$ and*

$$f \circ \theta = \pi_2$$

*holds in a neighbourhood of the origin.*

*Proof.* Define

$$g\colon \mathbb{R}^{k+\ell} \to \mathbb{R}^{k+\ell} \qquad \text{by} \qquad g(x, y) := \big(x, f(x, y)\big). \tag{3.3}$$

Then for the differential of $g$ we have

$$Dg(0) = \begin{pmatrix} id_{\mathbb{R}^k} & 0 \\ D_x f(0) & D_y f(0) \end{pmatrix} \qquad \Longleftrightarrow \qquad Dg(0)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ D_x f(0) u + D_y f(0) v \end{pmatrix},$$

where $u \in \mathbb{R}^k$ and $v \in \mathbb{R}^\ell$.

If $(u\ v) \in \ker Dg(0)$, then $u = 0$ and $D_y f(0)v = 0$. However, $D_y f$ is an isomorphism by assumption of this theorem, so that $v = 0$. Therefore, $Dg(0)$ is injective and, hence, an isomorphism.

By Theorem 2.24, there is a local inverse $\theta\colon \mathbb{R}^{k+\ell} \to \mathbb{R}^{k+\ell}$ to $g$, that is in a neighbourhood of the origin we have

$$g \circ \theta = id_{\mathbb{R}^{k+\ell}} \qquad \Longrightarrow \qquad \pi_2 = \pi_2 \circ \mathrm{id}_{\mathbb{R}^{k+\ell}} = \pi_2 \circ g \circ \theta = f \circ \theta.$$

Thus, the theorem is proved. □

**Corollary 3.4** (The implicit function theorem)**.** *Suppose that the assumptions of Theorem 3.2 hold. Then there exists a neighbourhood $V_1$ of $0 \in \mathbb{R}^k$, a neighbourhood $V_2$ of $0 \in \mathbb{R}^\ell$, and a unique smooth map $h\colon V_1 \to V_2$ such that*

$$f(x,y) = 0 \quad \Longleftrightarrow \quad y = h(x) \tag{3.5}$$

*whenever $(x,y) \in V_1 \times V_2$.*

*Furthermore, denoting $W := f^{-1}(0) \cap V_1 \times V_2$, the map*

$$\psi := \pi_1\big|_W \colon W \to V_1, \qquad (x,y) \mapsto x$$

*is a homemorphism, that is $(W,\psi)$ is a chart on $f^{-1}(0) \cap U$.*

*Proof.* Let $\theta\colon U \to \theta(U)$ be the local diffeomorphism provided by Theorem 3.2. Pick any open subsets $V_1$ and $V_2$ as in the formulation of the theorem such that $V_1 \times V_2 \subset U$. For $x \in V_1$ define $h(x) := \pi_2 \circ \theta\,(x,0)$. Furthermore, for $(x,y) \in V_1 \times V_2$, denote

$$(z,w) := \theta^{-1}(x,y) = g(x,y) = \big(x, f(x,y)\big).$$

Here we used the fact, that $g$, which is given by (3.3), is the inverse of $\theta$. Then

$$\begin{aligned} f(x,y) = 0 \quad &\Longrightarrow \quad 0 = f \circ \theta \circ \theta^{-1}(x,y) = f \circ \theta\,(z,w) = w \\ &\Longrightarrow \quad (z,0) = \big(x, f(x,y)\big). \end{aligned}$$

Hence, $z = x$ and $(x,y) = \theta(x,0)$, which yields in turn $y = h(x)$.

Furthermore, for any $x \in V_1$ we have

$$(x,0) = g \circ \theta\,(x,0) = g\big(\pi_1 \circ \theta\,(x,0),\ \pi_2 \circ \theta(x,0)\big)$$

From the definition of $g$ we obtain $x = \pi_1 \circ \theta\,(x,0)$ and, hence, $0 = f\big(x, h(x)\big)$.

To show the uniqueness, notice that

$$\begin{aligned} f\big(x,\ h(x)\big) = 0 \quad &\Longrightarrow \quad g\big(x, h(x)\big) = \big(x, f\big(x, h(x)\big)\big) = (x,0) \\ f\big(x,\ \hat h(x)\big) = 0 \quad &\Longrightarrow \quad g\big(x, \hat h(x)\big) = (x,0). \end{aligned}$$

Since $g$ is a local diffeomorphism, we obtain $h(x) = \hat h(x)$ provided $x$ is sufficiently close to the origin.

Furthermore, notice that the map

$$V_1 \to W, \qquad x \mapsto \big(x, h(x)\big)$$

is a continuous inverse of $\psi$. Hence, $\psi$ is a homeomorphism. □

*Remark* 3.6. The hypothesis of Corollary 3.4 implies that the differential of $f$ at the origin is *surjective*. In fact, the surjectivity of the differential is decisive in Theorem 3.2 and Corollary 3.4, whereas the hypothesis that $D_y f(0)$ is an isomorphism can be achieved by a linear change of coordinates, see the proof of Theorem 3.10 below for some details.

The proofs of Theorem 3.2 and Corollary 3.4 show that there is a chart $(U, \varphi)$ on $\mathbb{R}^{k+\ell}$ such that $\varphi\big(U \cap N\big) \subset \mathbb{R}^k \times \{0\}$. This motivates the following.

**Definition 3.7.** Let $N$ be a manifold of dimension $k + \ell$. A subset $M \subset N$ is said to be *a submanifold* of dimension $k$ (or $k$*-subamnifold*), if for each point $m \in M$ there exists a chart $(U, \varphi)$ on $N$ centered at $m$ such that

$$\varphi\big(U \cap N\big) = \varphi(U) \cap \big(\mathbb{R}^k \times \{0\}\big) \tag{3.8}$$

holds. Under these circumstances, the chart $(U, \varphi)$ is said to be *adapted to* $M$.

Notice that if $(U, \varphi)$ is an adapted chart, then $(M \cap U,\ \psi)$ is a chart on $M$, where

$$\psi\colon = \pi_1 \circ \varphi\big|_{U\cap M} \colon U \cap M \to \mathbb{R}^k.$$

**Proposition 3.9.** *A $k$-submanifold is a smooth $k$-manifold.*

*Proof.* By its very definition, a $k$-submanifold is equipped with a $C^0$-atlas $\mathcal{U}$, consisting of restrictions of all adapted charts.

I claim that this atlas is in fact smooth. Indeed, let $(U_1, \varphi_1)$ and $(U_2, \varphi_2)$ be two charts adapted to $M$. Denoting by $\imath_1 \colon \mathbb{R}^k \to \mathbb{R}^{k+\ell}$ the inclusion $\imath_1(x) = (x, 0)$, we have

$$\psi_1 \circ \psi_2^{-1}(x) = \psi_1\big(\varphi^{-1}(x, 0)\big) = \pi_1 \circ \varphi_1 \circ \varphi_2^{-1} \circ \imath_1\ (x) = \pi_1 \circ \theta_{12} \circ \imath_1\ (x).$$

Thus, $\mathcal{U}$ is smooth. □

We are now in the position to state one of the central theorems of this chapter.

**Theorem 3.10.** *Let $M$ and $N$ be smooth manifolds. If $n$ is a regular value of a smooth map $f\colon M \to N$ and $\dim M \geq \dim N$, then $f^{-1}(n)$ is a submanifold of $M$ of dimension $k := \dim M - \dim N$.*

*Proof.* Denote

$$\ell = \dim N \qquad \Longrightarrow \qquad \dim M = k + \ell.$$

Pick any $m \in f^{-1}(n)$ and any charts $(U, \varphi)$ and $(V, \psi)$ centered at $m$ and $n$ respectively. Let $F = \psi \circ f \circ \varphi^{-1}$ be the coordinate representation of $f$ with respect to the charts $(U, \varphi)$ and $(V, \psi)$. Since $\varphi$ and $\psi$ are diffeomophisms, we obtain that the differential $F_*$ of $F$ is surjective at the origin (in fact, at any point from $F^{-1}(0)$). In particular, $\dim \ker F_*(0) = k$.

Choose a basis $(\mathrm{v}_1, \dots, \mathrm{v}_{k+\ell})$ of $\mathbb{R}^k$ such that $(\mathrm{v}_1, \dots, \mathrm{v}_k)$ is a basis of $\ker F_*(0)$. Set

$$A\colon \mathbb{R}^{k+\ell} \to \mathbb{R}^{k+\ell}, \qquad z \mapsto \sum_{j=1}^{k+\ell} z_j \mathrm{v}_j.$$

Notice that by the definition of $A$ and elementary facts from linear algebra, the following holds:

- $A$ is an isomorphism;
- $A \circ \imath_1 \colon \mathbb{R}^k \to \ker F_*(0)$ is an isomorphism;

- $F_*(0) \circ A \circ \imath_2 \colon \mathbb{R}^\ell \to \mathbb{R}^\ell$ is an isomorphism.

Furthermore, consider the map $G := F \circ A \colon \mathbb{R}^{k+\ell} \to \mathbb{R}^\ell$. By Exercise 2.64, we have

$$G_*(0) = F_*(0) \circ A \qquad \Longrightarrow \qquad D_y G = F_*(0) \circ A \circ \imath_2.$$

Since the letter map is an isomorphism, by the proofs of Theorem 3.2 and Corollary 3.4 we obtain a chart $(W, \xi)$ on $\mathbb{R}^{k+\ell}$ adapted to $G^{-1}(0)$, that is

$$\xi\big(W \cap G^{-1}(0)\big) = \xi(W) \cap \big(\mathbb{R}^k \times \{0\}\big).$$

Without loss of generality we can assume that $W$ is contained in $A^{-1}\big(\varphi(U)\big)$.

Various charts involved in the proof are shown schematically on Figure 3.1.

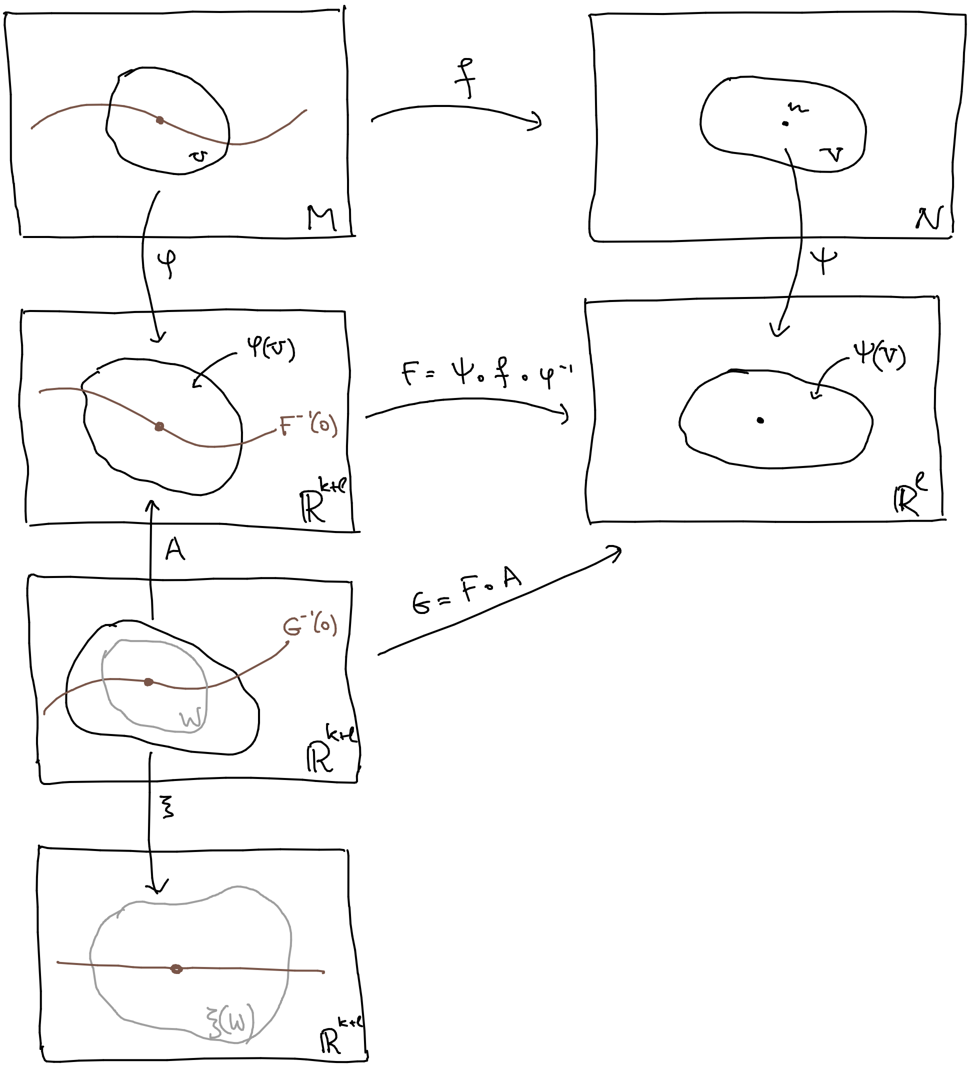


Figure 3.1: Scheme of the proof of Theorem 3.10.

Define a chart $(\hat{W}, \hat{\xi}\,)$ on $\mathbb{R}^{k+\ell}$ by

$$(\hat{W}, \hat{\xi}\,) = \big(\, A^{-1}(W),\ \xi \circ A^{-1}\big).$$

Since $z \in G^{-1}(0) \;\Leftrightarrow\; Az \in F^{-1}(0)$, we obtain

$$\hat{\xi}\big(\hat{W} \cap F^{-1}(0)\big) = \xi\big(W \cap G^{-1}(0)\big) = \xi(W) \cap \big(\mathbb{R}^k \times \{0\}\big).$$

Finally, setting

$$\varphi_1 := \hat{\xi} \circ \varphi \qquad \text{and} \qquad U_1 = \varphi_1^{-1}\big(\hat{\xi}(\hat{W})\big) = \varphi^{-1}\big(\hat{W}\big)$$

we obtain

$$\varphi_1\big(U_1 \cap f^{-1}(n)\big) = \hat{\xi}\big(\hat{W} \cap F^{-1}(0)\big) = \xi(W) \cap \big(\mathbb{R}^k \times \{0\}\big) = \varphi_1(U_1) \cap \big(\mathbb{R}^k \times \{0\}\big).$$

Thus, $(U_1, \varphi_1)$ is a chart adapted to $f^{-1}(n)$ at $m$. $\square$

Notice the following: If $\dim M < \dim N$, then $n$ is a regular value of smooth map $f\colon M \to N$ if and only if $n \notin \operatorname{Im} f$, see the paragraph following Definition 2.28. In this case $f^{-1}(n) = \varnothing$ is also (by definition) a smooth manifold. Thus, the condition $\dim M \geq \dim N$ can be dropped in the formulation of Theorem 3.10.

**Proposition 3.11.** *In the setting of Theorem 3.10, for any $m \in f^{-1}(n)$ we have*

$$T_m f^{-1}(n) = \ker f_*(m).$$

*Proof.* Pick any curve $\gamma$ in $f^{-1}(n)$ through $m$. Since $\gamma$ lies in the level set of $f$, we have

$$f \circ \gamma\,(t) = n \qquad \text{for all } t \in (-\varepsilon, \varepsilon). \tag{3.12}$$

Since the constant curve $t \mapsto n$ represents the zero vector in $T_nN$, by the definition of the differential of $f$ and (3.12) we obtain $f_*(m)\big([\gamma]\big) = 0$. In other words any vector $[\gamma]$ tangent to $f^{-1}(n)$ lies in the kernel of $f_*(m)$. $\square$

*Example* 3.13.

(i) Consider the map $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$, $f(x) = |x|^2$. Then $1$ is a regular value of $f$. In particular, $S^n = f^{-1}(1)$ is a manifold of dimension $n$. Of course, the reader knows this fact by now very well.

(ii) Let $M_n(\mathbb{R})$ be the space of all $n \times n$ matrices with real entries. One can show that $1$ is a regular value of the function $\det\colon M_n(\mathbb{R}) \to \mathbb{R}$, $A \mapsto \det A$. Consequently,

$$\mathrm{SL}_n(\mathbb{R}) := \big\{ A \in M_n(\mathbb{R}) \mid \det A = 1 \big\}$$

is a manifold of dimension $\dim M_n(\mathbb{R}) - 1 = n^2 - 1$.

Let us compute the tangent space to $\mathrm{SL}_n(\mathbb{R})$ at the point $\mathbb{1}$. To this end, it is convenient to identify $M_n(\mathbb{R})$ with $\mathbb{R}^{n^2}$. Recalling that

$$\det A = \sum_{\sigma} \operatorname{sign}\sigma \, a_{1\,\sigma(1)} \dots a_{n\,\sigma(n)},$$

where $\sigma$ runs through all permutations of the set $\{1, \dots, n\}$, for any $B \in M_n(\mathbb{R})$ we obtain

$$\begin{aligned}\det\big(\mathbb{1} + tB\big) &= (1 + t\, b_{11})(1 + t\, b_{22}) \dots (1 + t\, b_{nn}) \\ &\quad + \sum_{\sigma \neq id} \operatorname{sign}\sigma \,\big(\delta_{1\,\sigma(1)} + tb_{1\,\sigma(1)}\big)\big(\delta_{2\,\sigma(2)} + tb_{2\,\sigma(2)}\big) \dots \big(\delta_{n\,\sigma(n)} + tb_{n\,\sigma(n)}\big).\end{aligned}$$

Notice that for any $\sigma \neq id$, $\sigma(i) \neq i$ at least for two values of $i$. Hence, the last term in the above expression is $o(t)$. This yields

$$\det\big(\mathbb{1} + tB\big) = \big(1 + t\ \mathrm{tr}\, B + o(t)\big) + o(t).$$

Consequently, $\det_*(\mathbb{1})\, B = \mathrm{tr}\, B$ and therefore

$$T_{\mathbb{1}}\mathrm{SL}_n(\mathbb{R}) = \big\{ B \in M_n(\mathbb{R}) \mid \mathrm{tr}\, B = 0 \big\}.$$

(iii) Let $Sym^n(\mathbb{R}) \subset M_n(\mathbb{R})$ denote the subspace of all symmetric matrices. One can show that the identity matrix $\mathbb{1} \in Sym^n(\mathbb{R})$ is a regular value of the map

$$f\colon M_n(\mathbb{R}) \to Sym^n(\mathbb{R}), \qquad f(A) = A \cdot A^t. \tag{3.14}$$

Consequently,

$$O(n) := \big\{ A \in M_n(\mathbb{R}) \mid A \cdot A^t = \mathbb{1} \big\}$$

is a manifold and

$$\begin{aligned} \dim O(n) &= \dim M_n(\mathbb{R}) - \dim Sym^n(\mathbb{R}) = n^2 - \frac{n(n+1)}{2} \\ &= \frac{n(n-1)}{2}. \end{aligned}$$

Notice that if we would consider (3.14) as a map $M_n(\mathbb{R}) \to M_n(\mathbb{R})$, then $\mathbb{1}$ would not be a regular value.

Just like in the case of $\mathrm{SL}_n(\mathbb{R})$, let us compute the tangent space to $O(n)$ at the point $\mathbb{1}$. We have

$$f(\mathbb{1} + sB) = (\mathbb{1} + sB) \cdot (\mathbb{1} + sB)^t = \mathbb{1} + s\big(B + B^t\big) + o(s).$$

Hence, $f_*(\mathbb{1})B = B + B^t$ and

$$T_{\mathbb{1}}O(n) = \big\{ B \in M_n(\mathbb{R}) \mid B^t = -B \big\}.$$

We finish this section by Sard's theorem, which, loosely speaking, says that for any smooth map almost any point is a regular value. More precisely, we say that a subset $A$ of a smooth $k$-manifold $M$ is of measure zero, if for any chart $(U, \varphi)$ on $M$ the set $\varphi\big(A \cap U\big) \subset \mathbb{R}^k$ is of measure zero.

**Theorem 3.15** (Sard)**.** *Let $f\colon M \to N$ be a smooth map between smooth manifolds. Then almost any point $n \in N$ is a regular value of $f$, that is the set of critical values for $f$ is of measure zero.* □

A proof of Sard's theorem can be found for example in [BT03, 9.4] or [Mil65, §3]. L 5

## 3.2 Immersions and embeddings

Just like maps with surjective differentials can be conveniently described as projections after applying a diffeomorphisms, the maps with injective differentials admit an analogous description.

**Theorem 3.16.** *Let $U$ be an open subset of $\mathbb{R}^k$ containing the origin and $f\colon U \to \mathbb{R}^k \times \mathbb{R}^\ell$ be a smooth map such that $f(0) = 0$ and*

$$f_{1*}(0)\colon \mathbb{R}^k \to \mathbb{R}^k, \qquad \textit{where } \ f_1 := \pi_1 \circ f,$$

*is an isomorphism. Then there exists a neighbourhood $V \subset \mathbb{R}^{k+\ell}$ of the origin and a diffeomorphism $\theta\colon V \to \theta(V) \subset \mathbb{R}^{k+\ell}$ such that $\theta \circ f = \imath_1$ and $\theta\big(V \cap f(U)\big) = \theta(V) \cap \big(\mathbb{R}^k \times \{0\}\big)$.*

*Proof.* The proof of this theorem is similar to the proof of Theorem 3.2.

Thus, consider the map

$$F\colon U \times \mathbb{R}^\ell \to \mathbb{R}^{k+\ell} = \mathbb{R}^k \times \mathbb{R}^\ell, \qquad F(x,y) := f(x) + (0,\, y) = \big(f_1(x),\ f_2(x) + y\big).$$

The differential of this map

$$F_*(0) = \begin{pmatrix} f_{1*}(0) & 0 \\ f_{2*}(0) & \mathrm{id}_{\mathbb{R}^\ell} \end{pmatrix}$$

is an isomorphism. Hence, there exists a neighbourhood $V$ of the origin and a diffeomorphism $\theta\colon V \to \theta(V)$ such that

$$\theta \circ F = \mathrm{id}_{\theta(V)}.$$

In particular, for any $(x,0) \in \theta(V)$ the above equality yields:

$$\theta \circ F\,(x,0) = \theta \circ f(x) = \imath_1(x) \qquad \Longrightarrow \qquad \theta \circ f = \imath_1.$$

Hence, $\theta(V) \cap \big(\mathbb{R}^k \times \{0\}\big) \subset \theta\big(V \cap f(U)\big)$.

To show the converse inclusion, let $(x,y) \in \theta\big(V \cap f(U)\big)$. Hence, there exists some $(z,w) \in V \cap f(U)$ such that $(x,y) = \theta(z,w)$. In this case we must have $(z,w) = f(x)$ for some $x \in U$ and therefore

$$(x,y) = \theta(z,w) = \theta \circ f(x) = (x,0).$$

Thus, $y = 0$ and $(x,0) \in V$, which yields $\theta\big(V \cap f(U)\big) \subset \theta(V) \cap \big(\mathbb{R}^k \times \{0\}\big)$. $\square$

**Definition 3.17.** A smooth map $f\colon M^k \to N^\ell$ such that $f_*(m)$ is injective at each point $m \in M$ is called *an immersion*. An immersion, which is a diffeomorphism onto a $k$-submanifold of $N$, is called *an embedding*.

Clearly, an immersion of $f\colon M \to N$ can exists only if $\dim M \leq \dim N$. Notice also, that by Theorem 3.16 each immersion is locally injective, however an immersion does not need to be globally injective. Even if an immersion is injective, this may fail to be an embedding. This is shown schematically on Fugures 3.2 and 3.3 below. In particular, the image of an immersion does not need to be a submanifold.

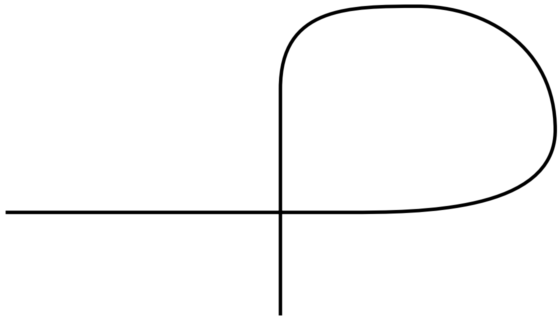

Figure 3.2: The image of a non-injective immersion $\mathbb{R} \to \mathbb{R}^2$.

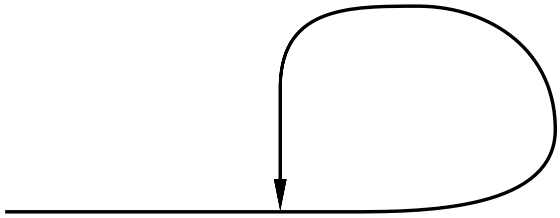

Figure 3.3: The image of an injective immersion $\mathbb{R} \to \mathbb{R}^2$, which is not an embedding.

**Proposition 3.18.** *An immersion which is a homeomorphism onto its image is an embedding.*

*Proof.* Denote $k := \dim M$ and $\ell := \dim N$. The proof consists of the following 3 steps.

**Step 1.** *For any $m \in M$ there exists a chart $(V, \psi)$ on $N$ centered at $n := f(m)$ with the following properties:*

- *$\psi_{1*}(n)\big|_{\operatorname{Im} f_*(m)} \colon \operatorname{Im} f_*(m) \to \mathbb{R}^k$ is an isomorphism, where $\psi_1 = \pi_1 \circ \psi$ and $\pi_1 \colon \mathbb{R}^\ell = \mathbb{R}^k \oplus \mathbb{R}^{\ell-k} \to \mathbb{R}^k$ is the projection.*
- *There exists a neighbourhood $U$ of $m$ such that $f(U) = V \cap f(M)$.*

Since $f$ is a homeomorphism onto its image, $f \colon M \to f(M)$ is an open map. In particular, for any open $\hat{U} \subset M$ there exists an open subset $\hat{V} \subset N$ such that $f(\hat{U}) = \hat{V} \cap f(M)$. If $\hat{U}$ is a neighbourhood of $m$, we can choose a chart $(V, \xi)$ centered at $n$ such that $V \subset \hat{V}$.

Furthermore, since $\xi_* \colon T_nN \to \mathbb{R}^\ell$ is an isomorphism and $\operatorname{Im} f_*(m)$ is a $k$-dimensional subspace of $T_nN$, we can find a linear isomorphism $A \colon \mathbb{R}^\ell \to \mathbb{R}^\ell$ such that

$$A\Big(\xi_*\big(\operatorname{Im} f_*(n)\big)\Big) = \mathbb{R}^k \times \{0\}.$$

Then $(V, \psi) = (V,\ A \circ \xi)$ is the required chart. Also, setting $U := f^{-1}(V)$ we obtain $f(U) = V \cap f(M)$.

**Step 2.** *$f(M)$ is a submanifold of $N$.*

Pick any $m \in M$ and a chart $(U, \varphi)$ centered at $m$. Pick also a chart $(V, \psi)$ as in the previous step. Denote also $W := \psi(V) \subset \mathbb{R}^\ell$.

Let $F = \psi \circ f \circ \varphi^{-1}$ be the coordinate representation of $f$. Denoting $F_1 := \pi_1 \circ F \colon \mathbb{R}^k \to \mathbb{R}^k$, we have

$$F_{1*}(0) = \pi_{1*}(0) \circ F_*(0) = \pi_{1*}(0) \circ \psi_*(n) \circ f_*(m) \circ \varphi_*^{-1}(0) = \psi_{1*}(0) \circ f_*(m) \circ \varphi_*^{-1}(0)$$

Since $\varphi_*^{-1}(0)$ is an isomorphism, by Step 1 we obtain that $F_{1*}(0)$ is injective. Hence, $F_{1*}(0)$ is an isomorphism. Hence, by Theorem 3.16 we can find a diffeomorphism[1] $\theta \colon W \to \theta(W) \subset \mathbb{R}^\ell$ such that

$$\theta \circ F = \imath_1 \qquad \Longleftrightarrow \qquad \big(\theta \circ \psi\big) \circ f \circ \varphi^{-1} = \imath_1$$

[1] Without loss of generality we may assume that $V$ was chosen so that $\theta$ is defined everywhere on $W$.

Denote $\hat{\psi} := \theta \circ \psi$. Then $(W, \hat{\psi})$ is a chart on $N$ adapted to $f(M)$.

**Step 3.** *$f$ is a diffeomorphism between $M$ and $f(M)$.*

Let $(U, \varphi)$ and $(W, \hat{\psi})$ be as in the preceding step. By the construction of $\hat{\psi}$, the coordinate representation of $f$ is $\hat{\psi} \circ f \circ \varphi^{-1} = \imath_1 \colon \mathbb{R}^k \to \mathbb{R}^\ell$. Since the restriction of $\pi_1 \circ \psi$ to $f(M) \cap W$ is a chart on $f(M)$, the coordinate representation of $f$ viewed as a map $f \colon M \to f(M)$ is given by

$$\pi_1 \circ \hat{\psi} \circ f \circ \varphi^{-1} = \pi_1 \circ \imath_1 = \mathrm{id}.$$

Hence, $f$ is a local diffeomorphism. Since $f \colon M \to f(M)$ is bijective, this is a diffeomorphism. □

**Corollary 3.19.** *If $M$ is compact, then any injective immersion $f \colon M \to N$ is an embedding.*

*Proof.* Pick a closed subset $A \subset M$. Since $A$ is closed in $M$, $A$ is compact and therefore $f(A)$ is compact in $N$. Since $N$ is Hausdorff, $f(A)$ is closed. Hence, $f$ is a closed map, i.e., the image of any closed subset is closed. This means that $f^{-1} \colon f(M) \to M$ is continuous, that is, $f \colon M \to f(M)$ is a homeomorphism. The statement of this corollary now follows immediately from Proposition 3.18. □

Theorem 3.10 combined with Sard's theorem allows us to construct many smooth manifolds, which are in fact submanifolds of Euclidean spaces. It turns out that any smooth manifold can be realized as a submanifold of an Euclidean space.

**Theorem 3.20** (Whitney's embedding theorem)**.** *For any smooth manifold $M$ there is an embedding of $M$ into $\mathbb{R}^n$ for some $n \in \mathbb{N}$.*

*Proof.* We prove Whitney's embedding theorem only in the case when $M$ is compact.

For any $m \in M$ choose a chart $(U_m, \varphi_m)$. Pick also open neigbourhoods $W_m \subset V_m$ and a bump function $\rho_m$ such that the following holds:

- $\overline{V}_m \subset U_m$;
- $\rho_m\big|_{\overline{W}_m} \equiv 1$ and $\rho_m < 1$ outside of $\overline{W}_m$;
- $\rho_m$ vanishes outside of $V_m$.

Since $M$ is compact, there is a finite subset $\{m_1, \dots, m_p\}$ of $M$ such that $\{W_i\}$ cover all of $M$, where $W_i := W_{m_i}$. Consider each $\psi_i := \rho_i \cdot \varphi_i := \rho_{m_i} \cdot \varphi_{m_i}$ as a smooth map $M \to \mathbb{R}^k$ (extended by zero outside of $V_m$), where $k = \dim M$. Finally, define

$$f \colon M \to \mathbb{R}^{pk+p}, \qquad f(m) = \big(\psi_1(m), \dots, \psi_p(m), \rho_1(m), \dots, \rho_p(m)\big).$$

Clearly, $f$ is smooth. I claim that this map is also injective. Indeed, pick any two distinct points $m$ and $\hat{m}$. Without loss of generality, we can assume $m \in W_1$. If $\hat{m} \in \overline{W}_1$, then $\psi_1(m) = \varphi_1(m) \neq \varphi_1(\hat{m}) = \psi_1(\hat{m})$. If $\hat{m} \notin \overline{W}_1$, then $1 = \rho_1(m) \neq \rho_1(\hat{m})$, so that $f$ is injective indeed.

Furthermore, assuming $m \in W_1$ again, $\psi_{1*}(m) \colon T_m M \to \mathbb{R}^k$ is an isomorphism. Hence, $f_*(m) \colon T_m M \to \mathbb{R}^{kp+p}$ is injective at any $m \in M$. By Corollary 3.19, $f$ is an embedding. □

Whitney's embedding theorem shows that any (abstract) manifold $M$ can be thought of as a submanifold of an Euclidean space. In other words, we could have defined manifolds as the subspaces of Euclidean spaces admitting charts[2] at each point. Some authors do take this point of view arguing that this yields the same pool of examples. While it is true of course that this would yield the same pool of examples, manifolds often do not arise as subsets of Euclidean spaces. For example, the real projective space is not obviously contained in any Euclidean space (and it is even not so obvious how to construct an embedding). Even if one decides to work with the submanifolds only, one finds out pretty soon that certain useful constructions, for example taking quotients by group actions, are incompatible with this setting. More importantly, it is useful to distinguish "inner" properties of manifolds from those of an embedding. All these reasons led to the necessity to separate abstract manifolds from their particular realizations as submanifolds.

## 3.3 Partitions of unity

Let $f\colon M \to \mathbb{R}$ be a (continuous) map.

**Definition 3.21.** The set

$$\operatorname{supp} f := \overline{\big\{m \in M \mid f(m) \neq 0\big\}}$$

is called *the support* of $f$. In other words, $\operatorname{supp} f$ is the closure of the set, where $f$ does not vanish.

*Example* 3.22.

(i) For $f\colon \mathbb{R} \to \mathbb{R},\ f(x) = x$ we have $\operatorname{supp} f = \mathbb{R}$.

(ii) For the cut off function (2.54) we have $\operatorname{supp} \chi_r = (-\infty,\, 1 + r]$.

(iii) For the bump function $\hat\rho\colon \mathbb{R}^k \to \mathbb{R},\quad \hat\rho(x) = \chi_1(|x|)$ we have

$$\operatorname{supp} \hat\rho = \big\{x \in \mathbb{R}^k \mid |x| \le 2\,\big\}.$$

**Definition 3.23.** A family of functions $\big\{\rho_\alpha\colon M \to \mathbb{R}_{\ge 0} \mid \alpha \in A\,\big\}$, where $A$ is an index set, is called *a partition of unity*, if the following holds:

(i) The family $\{\operatorname{supp} \rho_\alpha \mid \alpha \in A\,\}$ is locally finite, that is for each $m \in M$ there exists a neighbourhood $W \ni m$ such that the set $\{\alpha \in A \mid W \cap \operatorname{supp} \rho_\alpha \neq \varnothing\,\}$ is finite;

(ii) For each $m \in M$ we have

$$\sum_{\alpha\in A} \rho_\alpha(m) = 1. \tag{3.24}$$

Furthermore, a partition of unity $\{\rho_\alpha\,\}$ is said to be *subordinate to a covering* $\{U_\beta \mid \beta \in B\,\}$, if for each $\alpha \in A$ there exists some $\beta = \beta(\alpha)$ such that $\operatorname{supp} \rho_\alpha \subset U_{\beta(\alpha)}$.

Notice that (i) implies that for each $m \in M$ the set $\{\alpha \mid \rho_\alpha(m) \neq 0\}$ is finite so that (3.24) is a finite sum.

Any manifold trivially admits a partition of unity consisting of a single constant function. To obtain a non-trivial example, consider $M = \mathbb{R}$ and the family $\big\{\hat\rho_j \mid j \in \mathbb{Z}\big\}$, where

$$\hat\rho_j\colon \mathbb{R} \to \mathbb{R}, \qquad \hat\rho_j(x) = \chi_1\big(|x - j|\big). \tag{3.25}$$

[2]More precisely, admitting adapted charts.

In particular, $\hat\rho_j$ equals 1 everywhere on the interval $[j-1, j+1]$ and $\operatorname{supp}\hat\rho_j = [j-2, j+2]$. Hence, the function
$$\hat\rho(x) := \sum_{j\in\mathbb{Z}} \hat\rho_j(x)$$
is well-defined and positive everywhere on $\mathbb{R}$. Therefore, $\{\rho_j := \hat\rho_j/\hat\rho \mid j \in \mathbb{Z}\}$ is a partition of unity on $\mathbb{R}$. This partition of unity is subordinate for example to the following coverings:
$$\{(j-3, j+3) \mid j \in \mathbb{Z}\} \qquad \text{and} \qquad \{(-j, j) \mid j \in \mathbb{N}\}.$$

L 6

**Theorem 3.26** (Existence of Partition of Unity)**.** *Given an open covering $\mathcal{U}$ on a manifold, there is a partition of unity subordinate to $\mathcal{U}$.*

*Proof.* We prove this theorem only under a simplifying hypothesis that the manifold $M$ under consideration is compact.

Thus, pick any point $m \in M$ and a set $U_{\beta(m)} \in \mathcal{U}$ containing $m$. By Proposition 2.55, there exist a neighbourhood $V_m \subset U_{\beta(m)}$ and a bump function $\hat\rho_m$ such that $\hat\rho_m \equiv 1$ on $V_m$ and $\operatorname{supp}\hat\rho_m \subset U_{\beta(m)}$.

Since $M$ is compact, we can choose a finite subset $\{m_1, \dots, m_p\}$ such that $\{V_1, \dots, V_p\}$ is a covering of $M$, where $V_j := V_{m_j}$. Redenoting $\hat\rho_j := \hat\rho_{m_j}$, we obtain that
$$\hat\rho(m) := \sum_{j=1}^{p} \hat\rho_j(m)$$
is positive everywhere on $M$. Hence, $\{\rho_j := \hat\rho_j/\rho \mid j = \overline{1,p}\}$ is a partition of unity on $M$. Moreover, this is subordinate to $\mathcal{U}$, since $\operatorname{supp}\rho_j = \operatorname{supp}\hat\rho_j \subset U_{\beta(m_j)}$. $\square$

A couple of remarks are in place here. First, a proof of the above theorem in full generality can be found for example in [War83, 1.11] and uses the axiom of second countability, which we have not really used so far. This is one of the main reasons that the manifolds are required to be second countable.

Second, it is straightforward to generalize the definition of a smooth manifold to the complex holomorphic setting. Namely, we could have defined a complex manifold as a (Hausdorf second countable) topological space equipped with an atlas $\mathcal{U} := \{(U_\alpha, \varphi_\alpha) \mid \alpha \in A\}$, where $\varphi_\alpha \colon U_\alpha \to \varphi_\alpha(U) \subset \mathbb{C}^k$ is a homeomorphism, such that all transition maps
$$\theta_{\alpha\beta} := \varphi_\alpha \circ \varphi_\beta^{-1} \colon \mathbb{C}^k \to \mathbb{C}^k$$
are holomorphic. Most of the results we have seen so far are still valid in this setting (with an obvious replacement of the adjective "smooth" by "holomorphic"), except, most notably, Theorem 3.26 ( and related Theorem 3.20 and Proposition 2.55). The existence of the partition of unity on smooth manifolds and its non-existence on complex manifolds gives these two theories somewhat different flavours.

A typical application of the partition of unity is to existence questions, which we illustrate on the following result.

**Theorem 3.27.** *Let $M^k \subset N^\ell$ be an embedded submanifold. Then for any $h \in C^\infty(N)$ the restriction of $h$ to $M$ is a smooth function on $M$. Conversely, any smooth function $f$ on $M$ admits a smooth extension to $N$, that is there exists some $h \in C^\infty(N)$ such that $h\big|_M = f$.*

*Proof.* First notice that $h\big|_M = h \circ \imath_M$, where $\imath_M \colon M \to N$ is the embedding. The smoothness of the restriction then follows from the smoothness of $\imath_M$.

Thus, given $f \in C^\infty(M)$ we wish to find some $h \in \mathbb{C}^\infty(M)$ such that $h\big|_M = f$. Let $\mathcal{U}$ be a covering of $M$ by adapted charts. For the covering $\mathcal{U}_N := \mathcal{U} \cup \{N \setminus M\}$ of $N$ pick a partition of unity $\{\rho_\alpha\}$ on $N$ adapted to $\mathcal{U}_N$.

For each $\alpha$ define $f_\alpha \in C^\infty(M)$ by

$$f_\alpha(m) := \rho_\alpha(m) \cdot f(m) \qquad \Longrightarrow \qquad f = \sum_\alpha f_\alpha,$$

where at each point $m \in M$ only finitely many $f_\alpha$ are not vanishing.

I claim that each $f_\alpha$ admits an extension $h_\alpha$. First recall that if $(U, \varphi)$ is an adapted chart, then $(U \cap M,\, \psi)$ is a chart on $M$, where $\psi = \pi_1 \circ \varphi\big|_M$. Denoting by $F = f \circ \psi^{-1}$ the the coordinate representation of $f$, define

$$H_\alpha(x, y) := F(x) \cdot \big(\rho_\alpha \circ \varphi^{-1}\big)(x, y) \qquad \text{and} \qquad h_\alpha(n) := \begin{cases} H_\alpha \circ \varphi\,(n) & \text{if } n \in U, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $H_\alpha$ is a smooth function on $\mathbb{R}^\ell$, which vanishes outside of $\varphi(U)$ so that $h_a$ is well-defined and smooth. Moreover, the family $\{\operatorname{supp} h_\alpha \mid \alpha \in A\}$ is locally finite, since

$$\big(\, h_\alpha(n) \neq 0 \implies \rho_\alpha(n) \neq 0\,\big) \qquad \Longrightarrow \qquad \operatorname{supp} h_\alpha \subset \operatorname{supp} \rho_\alpha.$$

Therefore, we can define $h(n) := \sum_\alpha h_\alpha(n)$, which is smooth, because in a neighbourhood of each point $h$ is a finite sum of smooth functions. For $m \in M$, we have

$$h(m) = \sum_\alpha h_\alpha(m) = \sum_\alpha f_\alpha(m) = f(m).$$

Hence, $h$ is a smooth extension of $f$. $\square$

# Chapter 4

# The tangent bundle and the group of diffeomorphisms

## 4.1 Some elements of linear algebra

In what follows it may be useful to recall the following elements of linear algebra. Let $V$ be a linear vector space of dimension $k$. Any basis $\mathrm{v} = (\mathrm{v}_1, \dots, \mathrm{v}_k)$ of $V$ gives rise to the linear isomorphism

$$\mathbb{R}^k \to V, \qquad y \mapsto \sum_{j=1}^{k} y_j \mathrm{v}_j = \mathrm{v} \cdot y. \tag{4.1}$$

The very last term is a shortcut for the sum of the products on the left hand side and should be understood as a matrix multiplication, where the first "matrix" $\mathrm{v}$ consists of 1 row and $k$ columns (and its entries are vectors from $V$), whereas $y$ is a matrix with $k$ rows and 1 column, i.e., $y$ is just a column-vector.

Conversely, given a linear isomorphism $\mathbb{R}^k \to V$ we can construct a basis of $V$ just by taking the image of the standard basis of $\mathbb{R}^k$. This yields a bijective correspondence between the set of all bases of $V$ and the set of all isomorphisms $\mathbb{R}^k \to V$.

Furthermore, let $\mathrm{w} = (\mathrm{w}_1, \dots, \mathrm{w}_k)$ be another basis of $V$. Then $\mathrm{w}$ and $\mathrm{v}$ are related by the so called change-of-basis matrix $B$, which is obtained as follows. Decompose $\mathrm{w}_i$ in terms of the basis $\mathrm{v}$, that is write

$$\mathrm{w}_i = \sum_{j=1}^{k} b_{ij} \mathrm{v}_j.$$

Then $B = \big(b_{ij}\big)$ is the change-of-basis matrix between $\mathrm{w}$ and $\mathrm{v}$. In terms of the matrix multiplication used above, the relation between $\mathrm{v}$, $\mathrm{w}$, and $B$ can be elegantly expressed as follows:

$$\mathrm{w} = \mathrm{v} \cdot B.$$

Just to familiarize ourselves better with these notations, let $\varphi\colon V \to V$ be a linear map. The reader knows that given a basis of $V$, say $\mathrm{v}$, $\varphi$ can be represented by a $k \times k$-matrix $A$. This means, that if $(y_1, \dots, y_k)$ are coordinates of a vector $v \in V$, then the coordinates of $\varphi(v)$ are given by $A \cdot y$ (in this formula $y$ is interpreted as a column-vector). An elementary computation yields that the *jth column* of $A$ consists of coordinates of $\varphi(\mathrm{v}_j)$ with respect to $\mathrm{v}$. In other words, $A$ can be characterized by the equality $\varphi(\mathrm{v}) = \mathrm{v} \cdot A$. If $\mathrm{w} = \mathrm{v} \cdot B$ is another basis of $V$ as above, then we have

$$\varphi(\mathrm{w}) = \varphi\big(\mathrm{v} \cdot B\big) = \varphi(\mathrm{v}) \cdot B = \mathrm{v} \cdot A\,B = \mathrm{w} \cdot B^{-1} A\,B.$$

This yields that the matrix of $\varphi$ with respect to the basis w is $B^{-1}A\,B$. The reader surely knows this fact from the linear algebra, however the typical proof of this boils down to a tedious computation.

## 4.2 The tangent bundle

Consider the set

$$TM := \bigsqcup_{m\in M} T_mM,$$

which comes equipped with a map

$$\pi\colon TM \to M, \qquad \pi(\mathrm{v}) = m \;\Leftrightarrow\; \mathrm{v}\in T_mM.$$

For example, in the case $M=\mathbb{R}^k$ we can identify $T_m\mathbb{R}^k$ with $\mathbb{R}^k$ just as in (2.49) so that

$$T\mathbb{R}^k = \bigsqcup_{m\in\mathbb{R}^k} T_m\mathbb{R}^k = \mathbb{R}^k\times\mathbb{R}^k \qquad \text{and} \qquad \pi=\pi_1.$$

Furthermore, let $(U,\varphi)$ be a chart on $M$. Write $\varphi=(x_1,\dots,x_k)$ and recall that a each point $m\in M$ we have constructed the following basis of $T_mM$:

$$\partial_x\big|_m := \big(\partial_1,\dots,\partial_k\big)\big|_m, \qquad \text{where} \quad \partial_j\big|_m f = \frac{\partial}{\partial x_j}\Big|_{x=\varphi(m)}\Big(f\circ\varphi^{-1}(x)\Big). \tag{4.2}$$

See (2.48) and the proof of Proposition 2.47 for further details. Therefore, we obtain the bijection

$$U\times\mathbb{R}^k \to \pi^{-1}(U) = \bigsqcup_{m\in U} T_mM, \qquad (m,y)\mapsto \sum_{j=1}^k y_j\,\partial_j\big|_m.$$

Combining this with $\varphi\colon U\to\varphi(U)$, which is also a bijection, we obtain finally a bijective map

$$\tau=\tau_\varphi\colon \varphi(U)\times\mathbb{R}^k \to \pi^{-1}(U), \qquad \tau(x,y) := \sum_{j=1}^k y_j\,\partial_j|_{\varphi^{-1}(x)}.$$

**Proposition 4.3.** *Let $\mathcal{U} := \big\{(U_\alpha,\varphi_\alpha)\mid \alpha\in A\big\}$ be a smooth atlas on $M$. There is a unique second countable and Hausdorff topology on $TM$ such that*

$$\mathcal{V} := \Big\{\big(\pi^{-1}(U_\alpha),\ \tau_\alpha^{-1}\big)\mid \alpha\in A\Big\}$$

*is a $C^0$-atlas on $TM$, where $\tau_\alpha := \tau_{\varphi_\alpha}$. This atlas is in fact smooth so that $TM$ is a smooth manifold of dimension $2k$. Moreover, $\pi$ is a smooth map with surjective differential at each point.*

*Proof.* The proof consists of the following steps.

**Step 1.** *Write $\varphi_\alpha=(s_1,\dots,s_k)$ and $\varphi_\beta=(x_1,\dots,x_k)$. At each point $m\in U_\alpha\cap U_\beta$ we obtain two bases of $T_mM$, namely $\partial_s\big|_m$ and $\partial_x\big|_m$. Then the change-of-basis matrix between these two bases is the Jacobi-matrix of $\theta_{\alpha\beta}$, i.e.,*

$$\partial_x\big|_m = \partial_s\big|_m\cdot B \qquad \Longrightarrow \qquad B = D\theta_{\alpha\beta}. \tag{4.4}$$

Pick any function $f$ and let $F_\alpha := f \circ \varphi_\alpha^{-1}$ be the coordinate representation of $f$ with respect to the chart $\varphi_\alpha$. Then we have

$$F_\beta = f \circ \varphi_\beta^{-1} = f \circ \varphi_\alpha^{-1} \circ \varphi_\alpha \circ \varphi_\beta^{-1} = F_\alpha \circ \theta_{\alpha\beta}.$$

Therefore, using the above equality we obtain

$$\partial_j^x f = \frac{\partial F_\beta}{\partial x_j} = \sum_{i=1}^k \frac{\partial F_\alpha}{\partial x_i}\,\frac{\partial \theta_{\alpha\beta,i}}{\partial x_j} = \sum_{i=1}^k \frac{\partial \theta_{\alpha\beta,i}}{\partial x_j}\,\partial_i^s f, \tag{4.5}$$

where we dropped the point where the derivatives are computed from the notations. Since $f$ is an arbitrary function, we obtain (4.4).

**Step 2.** *For the coordinate transformation $\Theta_{\alpha\beta} := \tau_\alpha^{-1} \circ \tau_\beta$ on $TM$ we have*

$$\Theta_{\alpha\beta}(x, y) = \big(\,\theta_{\alpha\beta}(x),\ \theta_{\alpha\beta\,*}(x)\,y\,\big). \tag{4.6}$$

*In particular, $\Theta_{\alpha\beta}$ is smooth.*

Denote $\tau_\beta(x, y) = \mathrm{v}$. This means

$$\varphi_\beta\big(\pi(\mathrm{v})\big) = x \quad \text{and} \quad \mathrm{v} = \sum_{j=1}^k y_j\,\partial_j^x = \partial_x \cdot y,$$

where the right hand side of this equality is interpreted in the sense of (4.1). Therefore, by (4.4) we have

$$\mathrm{v} = \partial_x \cdot y = \partial_s \cdot D\theta_{\alpha\beta} \cdot y.$$

Denoting $\tau_\alpha^{-1}(\mathrm{v}) = (s, t)$, we have

$$s = \varphi_\alpha\big(\pi(\mathrm{v})\big) = \varphi_\alpha\big(\varphi_\beta^{-1}(x)\big) = \theta_{\alpha\beta}(x) \qquad \text{and} \qquad t = D\theta_{\alpha\beta} \cdot y.$$

**Step 3.** *We construct a topology on $TM$.*

Declare a set $V \subset TM$ to be open in $TM$ if and only if $\tau_\alpha^{-1}(V) = \tau_\alpha(V \cap U_\alpha)$ is open in $\mathbb{R}^{2k}$ for any $\alpha \in A$. We have

(i) $\varnothing$ is open and $\tau_\alpha^{-1}(M) = \varphi_\alpha(U_\alpha) \times \mathbb{R}^k \implies M$ is open.

(ii) $V_1$ and $V_2$ are open $\implies \tau_\alpha^{-1}\big(V_1 \cap V_2\big) = \tau_\alpha^{-1}(V_1) \cap \tau_\alpha^{-1}(V_2)$ is open.

(iii) If $V_\beta$ are open for each $\beta \in B$, then $\tau_\alpha^{-1}\big(\cup_\beta V_\beta\big) = \cup_\beta \tau_\alpha^{-1}(V_\beta)$ is open.

Hence, this yields a topology on $TM$ such that $\pi$ is continuous. Moreover, each $\big(\pi^{-1}(U_\alpha),\ \tau_\alpha^{-1}\big)$ is a chart on $TM$.

**Step 4.** *The topology of $TM$ is Hausdorff and second countable.*

Indeed, pick any two distinct points $\mathrm{v}_1, \mathrm{v}_2$ and consider the following cases:

(a) $\pi(\mathrm{v}_1) \neq \pi(\mathrm{v}_2) \implies \pi^{-1}(U_1)$ and $\pi^{-1}(U_2)$ separate $\mathrm{v}_1$ and $\mathrm{v}_2$ provided $U_1 \cap U_2 = \varnothing$.

(b) $\pi(\mathrm{v}_1) = \pi(\mathrm{v}_2) =: m \implies$ for any chart $(U_\alpha, \varphi_\alpha)$ containing $m$ the sets $\tau_\alpha\big(U_\alpha \times V_1\big)$ and $\tau_\alpha(U_\alpha \times V_2)$ are open and separate $\mathrm{v}_1$ and $\mathrm{v}_2$ provided $V_1, V_2 \subset \mathbb{R}^k$ separate $\pi_2\big(\tau_\alpha^{-1}(\mathrm{v}_1)\big)$ and $\pi_2\big(\tau_\alpha^{-1}(\mathrm{v}_2)\big)$.

Furthermore, the constructed topology is second countable for the following reason: Let $U_i$ be a countable basis of the topology of $M$ and $V_j$ be a countable basis for $\mathbb{R}^k$. Without loss of generality we can assume that each $U_i$ is contained in some chart $U_{\alpha_i}$. Then the collection of all sets of the form

$$\tau_{\alpha_i}\big(\varphi_{\alpha_i}(U_i) \times V_j\big)$$

is a countable basis for the topology of $TM$.

To finish the proof, pick a chart $\big(\pi^{-1}(U_\alpha),\, \tau_\alpha^{-1}\big)$ and consider the coordinate representation of $\pi$ with respect to this chart on $TM$ and the chart $(U_\alpha, \varphi_\alpha)$ on $M$:

$$\varphi_\alpha \circ \pi \circ \tau_\alpha\,(x, y) = \varphi_\alpha\big(\varphi_\alpha^{-1}(x)\big) = x \qquad \Longrightarrow \qquad \varphi_\alpha \circ \pi \circ \tau_\alpha = \pi_1.$$

Thus $\pi$ is smooth and its differential is surjective at each point. $\square$

To get a better understanding of the tangent bundle let us consider the case of an embedded submanifold $M \subset \mathbb{R}^\ell$. If $\imath\colon M \to \mathbb{R}^\ell$ denotes the embedding, for each $m \in M$ the map $\imath_*(m)\colon T_mM \to \mathbb{R}^\ell$ is injective. Consider the map

$$j\colon TM \to T\mathbb{R}^\ell = \mathbb{R}^\ell \times \mathbb{R}^\ell, \qquad j(\mathrm{v}) = \big(\imath(m),\, \imath_*(m)\mathrm{v}\big), \quad \text{where } m := \pi(\mathrm{v}).$$

Clearly, $j$ is injective.

Let $(U, \varphi)$ be a chart on $\mathbb{R}^\ell$ adapted to $M$. Thinking of $\mathbb{R}^\ell \times \mathbb{R}^\ell$ as the tangent bundle of $\mathbb{R}^\ell$, we obtain a chart $\big(\pi_1^{-1}(U),\, \tau_\varphi\big) = \big(U \times \mathbb{R}^\ell,\, \tau_\varphi\big)$ on $\mathbb{R}^\ell \times \mathbb{R}^\ell$.

Furthermore, recall that $m \in U$ belongs to $M$ if and only if $m = \varphi^{-1}(x, 0)$ for some $x \in \mathbb{R}^k$. Also, $\mathrm{v} \in T_m\mathbb{R}^\ell$ belongs to $T_mM$ if an only if

$$\mathrm{v} = \sum_{j=1}^{\mathbf{k}} y_j\, \partial_j\big|_{\psi^{-1}(x)} \qquad \Longleftrightarrow \qquad \tau_\varphi^{-1}(\mathrm{v}) = \big((x, 0),\, (y, 0)\big)$$

(notice that the summation runs to $k$!). Hence, $\big(U \times \mathbb{R}^\ell,\, \tau_\varphi\big)$ is a chart adapted to $j(TM)$ (up to a permutation of coordinates). In particular, $j(TM)$ is a $2k$-submanifold of $\mathbb{R}^{2\ell}$.

Finally, notice that $j$ is just the differential of $\imath$. Here we interpret the differential of $\imath$ as a map $TM \to T\mathbb{R}^\ell$.

**Exercise 4.7.** Recall that the restriction of $\psi := \pi_1 \circ \varphi$ is a chart on $M$, where $\pi_1\colon \mathbb{R}^\ell = \mathbb{R}^k \times \mathbb{R}^{\ell-k} \to \mathbb{R}^k$ is the projection. Show that the coordinate representation of $j$ with respect to the charts $\big(\pi^{-1}(U \cap M),\, \tau_\psi^{-1}\big)$ and $(U \times \mathbb{R}^\ell,\, \tau_\varphi^{-1})$ is the map

$$\mathbb{R}^k \times \mathbb{R}^k \ni (x, y) \mapsto \big(\imath_1(x),\, \imath_1(y)\big).$$

Deduce that $j$ is an immersion and, hence, an embedding.

Thus, we can simply identify $TM$ with $j(TM)$ so that

$$TM = \big\{(u, v) \in \mathbb{R}^{2\ell} \mid u \in M \text{ and } v \in T_uM\big\}.$$

*Example* 4.8. For $M = S^k \subset \mathbb{R}^{k+1}$, we have

$$TS^k = \big\{(x, y) \in S^k \times \mathbb{R}^{k+1} \mid \langle x, y\rangle = 0\big\} \subset \mathbb{R}^{2k+2}.$$

In particular, for $k = 1$ we obtain that $TS^1$ is a 2-submanifold of $\mathbb{R}^4$. In fact, we can realize $TS^1$ as a submanifold of $\mathbb{R}^3$ in the following sense. Consider the map

$$f\colon S^1 \times \mathbb{R} \to \mathbb{R}^4, \qquad f(x_0, x_1; t) = (x_0, x_1, tx_1, -tx_0).$$

One can check that $f$ is a diffeomorphism between $S^1 \times \mathbb{R} \subset \mathbb{R}^3$ and $TS^1$ so that we can in fact identify $TS^1$ with an infinite cylinder. L 7

## 4.3 Vector fields and their integral curves

**Definition 4.9.** A smooth map $v\colon M \to TM$ such that

$$\pi \circ v = id_M \qquad \Longleftrightarrow \qquad v(m) \in T_m M$$

is called a (smooth) *vector field* on $M$.

For example, the map

$$v\colon S^3 \to \mathbb{R}^4, \qquad v(x) = \big(x,\ (x_1, -x_0, x_3, -x_2)\big)$$

is a (smooth) vector field on $S^3$. Since the first component of $v$ must be $x$ by the very definition of the vector field, usually one simply writes

$$v(x) = \big(x_1, -x_0, x_3, -x_2\big).$$

Denote

$$\mathfrak{X}(M) := \big\{ v\colon M \to TM \text{ is a vector field } \big\}.$$

Clearly, $\mathfrak{X}(M)$ is a real vector space with respect to the following operations:

- $\big(v_1 + v_2\big)(m) := v_1(m) + v_2(m)$, where $v_1, v_2 \in \mathfrak{X}(M)$;
- $\big(\lambda v\big)(m) = \lambda v(m)$, where $v \in \mathfrak{X}(M)$ and $\lambda \in \mathbb{R}$.

In fact, any vector field can be multiplied by any smooth function:

$$\big(f \cdot v\big)(m) = f(m)v(m), \qquad \text{where } v \in \mathfrak{X}(M) \text{ and } f \in C^\infty(M).$$

We summarize this in the following.

**Proposition 4.10.** *The set $\mathfrak{X}(M)$ of all vector fields on $M$ has the structure of a module over $C^\infty(M)$ with respect to the pointwise addition and multiplication.* □

*Example* 4.11. Consider $M = \mathbb{R}^k$. We have seen that $T\mathbb{R}^k \cong \mathbb{R}^k \times \mathbb{R}^k$ and that the natural projection equals $\pi_1$. Hence, a vector field is a map of the form

$$v(x) = \big(x, y(x)\big),$$

where $y \in C^\infty(\mathbb{R}^k; \mathbb{R}^k)$. Hence, we can identify $\mathfrak{X}(\mathbb{R}^k)$ with $C^\infty(\mathbb{R}^k; \mathbb{R}^k)$ via the map

$$v = \big(id_{\mathbb{R}^k},\ y\big) \mapsto y.$$

More formally, this map is an isomorphism of $C^\infty(M)$-modules.

Generalizing the above example slightly, pick a chart $(U, \varphi)$ on a manifold $M$, where $\varphi = (x_1, \dots, x_k)$. Since (4.2) is a basis of $T_m M$, we can find the coordinates $\big(y_1(m), \dots, y_k(m)\big)$ of $v(m)$ with respect to this basis. In other words, $y\colon U \to \mathbb{R}^k$ is a map such that

$$v(m) = \partial_x\big|_m \cdot y(m)$$

holds at any point $m \in U$. Notice that the map $y$ is well defined even if $v$ is not necessarily smooth. This map is called *the coordinate (or local) representation of* $v$ with respect to the chart $(U, \varphi)$.

**Proposition 4.12.** *The map $v\colon M \to TM$ satisfying $\pi \circ v = id_M$ is a smooth vector field if an only if for each chart $(U, \varphi)$ as above the coordinate representation $y$ of $v$ is smooth.*

*Proof.* Recall that for any chart $(U, \varphi)$ on $M$ as above we constructed a chart $\big(\pi^{-1}(U),\ \tau_\varphi^{-1}\big)$ on $TM$. Just by the definitions of $\tau_\varphi$ and $y$, for the coordinate representation of $v$ with respect to these charts we have

$$\tau_\varphi^{-1} \circ v \circ \varphi^{-1} = \big(x,\ y \circ \varphi^{-1}(x)\big).$$

Hence, $v$ is smooth if and only if $y$ is smooth. $\square$

Thus, locally over each chart $U$ vector fields can be identified with smooth vector-valued maps just as in Example 4.11. It turns out, however, that in general no such identification can exist.[1]

Let $\gamma\colon (a, b) \to M$ be a smooth curve. At any point $t \in (a, b)$ we define *the tangent vector* $\dot\gamma(t) \in T_{\gamma(t)}M$ to $\gamma$ by

$$\dot\gamma(t)\,(f) := \frac{d}{dt}\Big(f \circ \gamma(t)\Big) \qquad \Longleftrightarrow \qquad \dot\gamma(t) := \big[\gamma(s+t)\big].$$

In the above equation the first definition yields a tangent vector as a derivation, whereas the second one as a class of curves through a point.

Consider $\mathbb{R}$ as a 1-dimensional manifold equipped with an atlas consisting of a single chart $(\mathbb{R}, \varphi)$, where $\varphi(t) = t$. Notice that for each fixed $t \in \mathbb{R}$ the derivation

$$\frac{d}{dt}\colon C^\infty(\mathbb{R}) \to \mathbb{R}, \qquad f \mapsto f'(t) \tag{4.13}$$

is non-trivial, since for each point $t$ there is a function $f$ such that $f'(t) \neq 0$. Hence, $\frac{d}{dt}$ is a basis of $T_t\mathbb{R}$.

**Proposition 4.14.** *We have*

$$\gamma_*(t)\left(\tfrac{d}{dt}\right) = \dot\gamma(t). \tag{4.15}$$

*Proof.* By (2.58), for any $f \in C^\infty(M)$ and any $t \in \mathbb{R}$ we have

$$\gamma_*(t)\left(\tfrac{d}{dt}\right) f = \big(f \circ \gamma\big)'(t) = \dot\gamma(t)(f).$$

Since $f$ is an arbitrary function, we obtain (4.15). $\square$

Let $M$ be a submanifold in $\mathbb{R}^\ell$. Denoting by $\imath$ the embedding, we can think of any curve $\gamma$ in $M$ as a curve in $\mathbb{R}^\ell$. More precisely, for any curve $\gamma\colon (a, b) \to M$, $\Gamma := \imath \circ \gamma$ is a curve in $\mathbb{R}^\ell$. We have

$$\dot\Gamma(t) = \big(\imath \circ \gamma\big)_* \tfrac{d}{dt} = \imath_* \circ \gamma_* \tfrac{d}{dt} = \imath_*\big(\dot\gamma(t)\big). \tag{4.16}$$

Here I omitted the points where the differential is computed at in the notations. In other words, thinking of $\imath_*$ as an identification between $T_mM$ and $\imath_*\big(T_m\big) \subset \mathbb{R}^\ell$, the tangent vector to $\gamma$ is just the ordinary tangent vector well known from the analysis course.

**Definition 4.17** (Integral curves)**.** A (smooth) curve $\gamma$ is called *an integral curve* of a vector field $v$ if

$$\dot\gamma(t) = v\big(\gamma(t)\big)$$

holds for any $t \in (a, b)$.

[1] If time permits, we will return to this below.

*Example* 4.18. Consider the curve $\gamma\colon \mathbb{R} \to S^3$, $\gamma(t) = \big(\sin t, \cos t, 0, 0\big)$. We have $\dot\gamma(t) = \big(\cos t, -\sin t, 0, 0\big)$. Furthermore, if $v$ is given by (4.3), then

$$v \circ \gamma\,(t) = \big(\cos t, -\sin t, 0, 0\big).$$

Hence, $\gamma$ is an integral curve of (4.3).

Let us consider integral curves on $\mathbb{R}^k$ in some detail. Thus, represent a vector field $v \in \mathfrak{X}(\mathbb{R}^k)$ by a smooth map $y\colon \mathbb{R}^k \to \mathbb{R}^k$ just as in Example 4.11 above. A map $\gamma\colon (a, b) \to \mathbb{R}^k$ is an integral curve of $v$ if and only if

$$\dot\gamma(t) = y\big(\gamma(t)\big) \qquad \Longleftrightarrow \qquad \begin{cases} \dot\gamma_1(t) = y_1\big(\gamma_1(t), \dots, \gamma_k(t)\big), \\ \qquad \cdots\cdots\cdots \\ \dot\gamma_k(t) = y_k\big(\gamma_1(t), \dots, \gamma_k(t)\big), \end{cases} \tag{4.19}$$

holds for any $t \in (a, b)$. In other words, an integral curve of a vector field is a solution of a system of ordinary differential equations (ODEs). Notice that the map $y$ does not depend on $t$, that is (4.19) is *an autonomous* system of ODEs.

Conversely, any system of ODEs as above, is uniquely specified by a map $y \in C^\infty(\mathbb{R}^k; \mathbb{R}^k)$. In view of Example 4.11, $y$ corresponds to a vector field $v$, whose integral curves are solutions of the initial system of ODEs. Thus, at least for Euclidean spaces, integral curves of vector fields and solutions of autonomous systems of ODEs are synonymous.

**Exercise 4.20.** Show that if $\gamma$ is a $C^1$-curve satisfying (4.19), then $\gamma$ is smooth.

Notice that for autonomous systems we have the following property: If $\gamma$ is a solution of (4.19) such that $\gamma(t_0) = m_0$, then for any $c \in (a, b)$

$$\gamma_c(t) := \gamma(t + c), \qquad t \in (a - c, b - c)$$

is also a solution. In other words, the integral curve $\gamma_1$ of $v$ such that $\gamma_1(t_1) = m_0$ satisfies

$$\gamma_1(t) = \gamma\big(t + t_0 - t_1\big),$$

that is $\gamma_1$ differs from $\gamma$ just by a shift of time. For this reason, one often chooses $t_0 = 0$ as the initial time for integral curves of vector fields.

By the main theorem of ODEs [Hal80, Sec.I.3], we obtain the following existence and uniqueness result.

**Theorem 4.21.** *Let $v$ be a smooth vector field on an open subset $\Omega \subset \mathbb{R}^k$. For any point $m_0 \in \Omega$ there exists a neighbourhood $V \subset \Omega$ of $m_0$ and a number $\varepsilon > 0$ with the following property: For any $m \in V$ there exists an integral curve*

$$\gamma = \gamma_m\colon (-\varepsilon, \varepsilon) \to \Omega \qquad \textit{such that} \qquad \gamma(0) = m.$$

*This integral curve is unique in the following sense: If $\beta\colon (-\delta, \delta) \to M$ is any other integral curve such that $\beta(0) = m$, then $\beta$ and $\gamma_m$ coincide on $(-\varepsilon, \varepsilon) \cap (-\delta, \delta)$. Moreover, the map*

$$\Phi\colon (-\varepsilon, \varepsilon) \times V \to \mathbb{R}^k, \qquad \Phi(t, m) := \gamma_m(t) \tag{4.22}$$

*is smooth.* □

**Definition 4.23.** An integral curve $\gamma\colon (a,b) \to M$ of a vector field $v$ is called *maximal*, if the following property holds: For any other integral curve $\beta\colon (c,d) \to M$ of $v$ such that for some $t_0 \in (a,b) \cap (c,d)$ we have $\gamma(t_0) = \beta(t_0)$, then:

(i) $(c,d) \subset (a,b)$;

(ii) $\beta = \gamma\big|_{(c,d)}$.

It is a well-known fact from the theory of ODEs, that for any $m_0 \in \mathbb{R}^k$ there is a unique maximal solution of (4.19) through $m_0$. A straightforward corollary is, that for any vector field $v$ on any manifold $M$ there is a unique maximal integral curve $\gamma$ of $v$ through a given point.

**Corollary 4.24.** *If $M$ is compact, then a maximal integral curve of any vector field is defined on all of $\mathbb{R}$.*

*Proof.* For each point $m \in M$ pick a chart $(U, \varphi)$ containing $m$. Writing $\varphi = (x_1, \dots, x_k)$, we obtain the coordinate representation of the vector field $v$ via the map $y\colon \Omega := \varphi(U) \to \mathbb{R}^k$. Then $\gamma\colon (a,b) \to U$ is an integral curve of $v$ if and only if for $\Gamma := \varphi \circ \gamma$ we have

$$\dot\Gamma(t) = y\big(\Gamma(t)\big) \qquad \text{for } \ t \in (a,b),$$

cf. (4.19). By Theorem 4.21, there exists a neighborhood $V_m$ such that for each $\hat m \in V_m$ the integral curve $\gamma_{\hat m}$ through $\hat m$ is defined on $(-\varepsilon_m, \varepsilon_m)$. By the compactness of $M$, we can find a finite collection of points $\{m_1, \dots, m_\ell\}$ such that the corresponding collection of neighbourhoods $\big\{V_j := V_{m_j} \mid 1 \le j \le \ell\big\}$ covers all of $M$. Set

$$\varepsilon := \frac{\min\{\varepsilon_{m_j} \mid 1 \le j \le \ell\}}{2} > 0.$$

Let $\gamma\colon (a,b) \to M$ be a maximal integral curve of $v$. Assuming $b < \infty$, the point $m_0 := \gamma(b-\varepsilon)$ lies in some $V_j$. By the construction of $\varepsilon$, there is a unique integral curve $\gamma_{m_0}$, which is well-defined on $(-2\varepsilon, 2\varepsilon)$ and satisfies $\gamma_{m_0}(0) = m_0$. Set

$$\hat\gamma\colon (a,\, b+\varepsilon) \to M, \qquad \hat\gamma(t) = \begin{cases} \gamma(t) & \text{for } t \in (a,\, b-\varepsilon), \\ \gamma_{m_0}\big(t - b + \varepsilon\big) & \text{for } t \in \big[b-\varepsilon,\, b+\varepsilon\big). \end{cases}$$

Notice that $\hat\gamma$ is continuous since $\gamma_{m_0}(b-\varepsilon) = m_0 = \gamma(b-\varepsilon)$. In fact, by the construction $\hat\gamma$ is an integral curve of $v$ on $(a, b-\varepsilon) \cup (b-\varepsilon, b+\varepsilon)$. It follows that $\hat\gamma$ is a $C^1$-integral curve of $v$ and therefore smooth by Exercise 4.20. Thus, $\hat\gamma$ is an integral curve of $v$ defined on a larger interval. This contradicts the maximality of $\gamma$. $\square$

**Exercise 4.25.** Modify the proof of Corollary 4.24 to show the following: For any manifold $M$ and any vector field $v$ such that

$$\operatorname{supp} v := \overline{\{m \mid v(m) \neq 0\}}$$

is compact, any maximal integral curve of $v$ is defined on all of $\mathbb{R}$.

## 4.4 Flows and 1-parameter groups of diffeomorphisms

In this section I assume that $M$ is a compact manifold.

For a vector field $v$ define *the flow* of $v$ to be the map

$$\Phi\colon \mathbb{R}\times M \to M, \qquad \Phi(t,m) = \gamma_m(t).$$

Of course, this is just the map $\Phi$ of Theorem 4.21 extended to the whole real line. Sometimes, (4.22) is referred to as *the local flow* of $v$.

Beside the flow, for each fixed $t\in\mathbb{R}$ it is also convenient to consider

$$\Phi_t\colon M\to M, \qquad \Phi_t(m) = \Phi(t,m) = \gamma_m(t).$$

**Proposition 4.26.** *The following holds:*

*(i)* *Each $\Phi_t$ is a diffeomorphism. Moreover, $\Phi_t^{-1} = \Phi_{-t}$;*

*(ii)* *For any $t,s\in\mathbb{R}$ we have $\Phi_t\circ\Phi_s = \Phi_{t+s} = \Phi_s\circ\Phi_t$;*

*(iii)* *$\Phi_0 = id_M$;*

*Proof.* For $m\in M$ and $t\in\mathbb{R}$ denote $\Phi_t(m) = \hat m$. This means that $\gamma_m(t) = \hat m$, where $\gamma_m$ is an integral curve of $v$ such that $\gamma_m(0) = m$.

Consider the curve $\beta$ defined by $\beta(s) = \gamma_m(s+t)$. Then $\beta$ is an integral curve of $v$ and $\beta(0) = \gamma_m(t) = \hat m$, that is $\beta = \gamma_{\hat m}$. Hence,

$$\Phi_s(\hat m) = \gamma_{\hat m}(s) = \beta(s) = \gamma_m(s+t) = \Phi_{s+t}(m) \qquad \Longleftrightarrow \qquad \Phi_s\circ\Phi_t = \Phi_{s+t}.$$

Since *(iii)* holds by the very definition of $\Phi_t$, by *(ii)* we obtain

$$\Phi_{-t}\circ\Phi_t = id_M = \Phi_t\circ\Phi_{-t}.$$

In particular, each $\Phi_t$ is a diffeomorphism and $\Phi_t^{-1} = \Phi_{-t}$ ∎

**Definition 4.27.** A 1-parameter group of diffeomorphisms is any smooth map $\Phi\colon\mathbb{R}\times M\to M$ such that Properties *(i)*–*(iii)* of Proposition 4.26 hold.

To explain the above definition, notice that the set

$$\mathrm{Diff}(M) := \big\{ f\colon M\to M \mid f \text{ is a diffeomorphism}\,\big\}$$

is a group with respect to the composition operation. $\mathrm{Diff}(M)$ is called *the diffeomorphism group* of $M$. With this understood, a 1-parameter group of diffeomorphisms is simply a homomorphism of groups

$$\mathbb{R}\to\mathrm{Diff}(M), \qquad t\mapsto\Phi_t$$

such that $\Phi_t(m) = \Phi(t,m)$ depends smoothly on $(t,m)$.

Thus, Proposition 4.26 states that each vector field on a compact manifold generates a 1-parameter group of diffeomorphisms. Conversely, it turns out that any 1-parameter group of diffeomorphisms generates a vector field in the following sense.

**Proposition 4.28.** *For any 1-parameter group of diffeomorphisms $\Phi$ there exists a vector field $v$, whose 1-parameter group of diffeomorphisms coincides with $\Phi$.*

*Proof.* For any $m \in M$ denote

$$\gamma_m \colon \mathbb{R} \to M, \qquad \gamma_m(t) := \Phi(t,m) \qquad \text{and} \qquad v(m) := \dot\gamma_m(0).$$

The reader should check that $v$ is a smooth vector field.

Furthermore, denote $\gamma_m(t) = \hat m$ and observe that

$$\gamma_{\hat m}(s) = \Phi_s(\hat m) = \Phi_s\big(\Phi_t(m)\big) = \Phi_{t+s}(m) = \gamma_m(t+s).$$

In other words, if $a_t \colon \mathbb{R} \to \mathbb{R}$ is defined by $a_t(s) = t+s$, then $\gamma_{\hat m} = \gamma_m \circ a_t$. Hence,

$$\begin{aligned} v\big(\gamma_m(t)\big) &= v\big(\hat m\big) = \gamma_{\hat m *}\big|_{s=0}\big(\tfrac{d}{ds}\big) = \big(\gamma_m \circ a_t\big)_*\big|_{s=0}\big(\tfrac{d}{ds}\big) \\ &= \gamma_{m*}\big|_{s=t} \circ \big(a_t\big)_*\big|_{s=0}\big(\tfrac{d}{ds}\big) = \gamma_{m*}\big|_{s=t}\big(\tfrac{d}{ds}\big) = \dot\gamma_m(t). \end{aligned}$$

Thus, $\gamma_m$ is the integral curve of $v$. Therefore, the 1-parameter group of diffeomorphisms generated by $v$ is

$$(t,m) \mapsto \gamma_m(t) = \Phi(t,m),$$

In other words, the 1-parameter group of diffeomorphisms generated by $v$ coincides with $\Phi$. $\square$

To sum up, for compact manifolds there is a natural bijective correspondence between vector fields and 1-parameter groups of diffeomorphisms. L 8

# Chapter 5

# Differential forms and the Brouwer degree

## 5.1 Some elements of (multi)linear algebra

For any vector space $V$ over $\mathbb{R}$ of dimension $k$, we can associate *the dual vector space*

$$V^* := \{\chi\colon V \to \mathbb{R} \mid \chi \text{ is linear}\}.$$

Moreover, if $\mathrm{v}$ is a basis of $V$, then

$$\mathrm{v}^* := \big(\mathrm{v}_1^*, \dots, \mathrm{v}_k^*\big) \qquad \text{uniquely determined by } \mathrm{v}_j^*(\mathrm{v}_i) = \delta_{ij}$$

is a basis of $V^*$. In particular, $\dim V^* = \dim V = k$.

In the case $V = \mathbb{R}^k$ we have a distinguished isomorphism

$$\mathbb{R}^k \to \big(\mathbb{R}^k\big)^*, \qquad y \mapsto \chi_y, \quad \chi_y(x) = \sum_{j=1}^k x_j y_j = \langle x, y\rangle \tag{5.1}$$

so that in practice we identify $\big(\mathbb{R}^k\big)^*$ with $\mathbb{R}^k$ via this isomorphism.

There are other ways to construct new vector spaces from a given one. Particularly relevant for us will be *the space of $p$-forms* on $V$, where $p \in \mathbb{N}$. This is denoted by $\Lambda^p V^*$ and consists of all maps $\omega\colon V \times \dots \times V \to \mathbb{R}$ such that the following holds:

(a) $\omega(\mathrm{w}_1, \dots, \mathrm{w}_j, \dots, \mathrm{w}_p)$ is linear in each argument;

(b) $\omega(\mathrm{w}_1, \dots, \mathrm{w}_j, \mathrm{w}_{j+1}, \dots, \mathrm{w}_p) = -\omega(\mathrm{w}_1, \dots, \mathrm{w}_{j+1}, \mathrm{w}_j, \dots, \mathrm{w}_p)$ for any $\mathrm{w}_1, \dots, \mathrm{w}_p \in V$.

In particular, for $p = 1$ the second condition above is vacuous so that $\Lambda^1 V^* = V^*$.

Elements of $\Lambda^p V^*$ are called $p$-forms on $V$.

Let us consider the case $p = 2$ in some details. Notice that we have a natural map

$$\begin{aligned} V^* \times V^* \to \Lambda^2 V^*, \qquad & (\chi_1, \chi_2) \mapsto \chi_1 \wedge \chi_2, \quad \text{where} \\ & \chi_1 \wedge \chi_2\big(\mathrm{w}_1, \mathrm{w}_2\big) = \chi_1(\mathrm{w}_1)\chi_2(\mathrm{w}_2) - \chi_1(\mathrm{w}_2)\chi_2(\mathrm{w}_1). \end{aligned}$$

Notice that this maps is skew-symmetric, that is $\chi_1 \wedge \chi_2 = -\chi_2 \wedge \chi_1$. In particular, $\chi \wedge \chi = 0$ for any $\chi \in V^*$.

**Proposition 5.2.** *If* $\mathrm{v}^*$ *is the dual basis of* $V^*$*, then*

$$\begin{array}{cccc} \mathrm{v}_1^* \wedge \mathrm{v}_2^*, & \mathrm{v}_1^* \wedge \mathrm{v}_3^*, & \dots, & \mathrm{v}_1^* \wedge \mathrm{v}_k^*, \\ & \mathrm{v}_2^* \wedge \mathrm{v}_3^*, & \dots, & \mathrm{v}_2^* \wedge \mathrm{v}_k^* \\ & \dots\dots & \dots\dots & \dots\dots \\ & & & \mathrm{v}_{k-1}^* \wedge \mathrm{v}_k^* \end{array} \tag{5.3}$$

*is a basis of* $\Lambda^2 V^*$*. In particular,* $\dim \Lambda^2 V^* = \frac{k(k-1)}{2}$.

*Proof.* Notice first that (5.3) consists of linearly independent 2-forms:

$$\sum_{i<j} \lambda_{ij} \mathrm{v}_i^* \wedge \mathrm{v}_j^* = 0 \qquad \Longrightarrow \qquad 0 = \Big( \sum_{i<j} \lambda_{ij} \mathrm{v}_i^* \wedge \mathrm{v}_j^* \Big) \big( \mathrm{v}_p, \mathrm{v}_q \big) = \lambda_{pq},$$

where $p < q$.

Furthermore, any 2-form $\omega$ on $V$ can be uniquely represented as a linear combination of (5.3), since

$$\lambda_{ij} := \omega(\mathrm{v}_i, \mathrm{v}_j) \qquad \Longrightarrow \qquad \omega = \sum_{i<j} \lambda_{ij}\, \mathrm{v}_i^* \wedge \mathrm{v}_j^*. \qquad \square$$

One more particularly important case arises when $p = k$. Define $\mathrm{v}_1^* \wedge \cdots \wedge \mathrm{v}_k^*$ as follows. Given any $k$-tuple $\mathrm{w} = \big( \mathrm{w}_1, \dots, \mathrm{w}_k \big)$ of vectors in $V$, decompose each $\mathrm{w}_j$ in terms of the basis $\mathrm{v}$, that is write $\mathrm{w}_j = \sum_i b_{ij} \mathrm{v}_i$. This yields a $k \times k$-matrix $B$ such that $\mathrm{w} = \mathrm{v} \cdot B$. Set

$$\mathrm{v}_1^* \wedge \cdots \wedge \mathrm{v}_k^* \, \big( \mathrm{w}_1, \dots, \mathrm{w}_k \big) = \det B.$$

Arguing just like in the case $p = 2$, we obtain that $\mathrm{v}_1^* \wedge \cdots \wedge \mathrm{v}_k^*$ is a basis of $\Lambda^k V^*$. In particular, $\dim \Lambda^k V^* = 1$.

For any linear map $A \colon V \to W$ between linear spaces we can associate *the dual map*

$$A^* \colon W^* \to V^*, \qquad \big( A^* \chi \big)(\mathrm{v}) = \chi \big( A\mathrm{v} \big).$$

Just by the very definition of the dual map, we have

$$\big( AB \big)^* = B^* A^* \tag{5.4}$$

for any linear maps $A : V \to W$ and $B \colon U \to V$.

Let $\mathrm{v} = (\mathrm{v}_1, \dots, \mathrm{v}_k)$ be a basis of $V$. Assume that $\mathrm{w} := A\mathrm{v} = \big( A\mathrm{v}_1, \dots, A\mathrm{v}_k \big)$ is a basis of $W$ so that $A$ is an isomorphism. Then $A^*$ maps $\mathrm{w}^*$ to $\mathrm{v}^*$. In particular, the dual of an isomorphism is itself an isomorphism.

In fact, for any $p \in \mathbb{N}$ and $A$ as above we have the corresponding map

$$A^* \colon \Lambda^p W^* \to \Lambda^p V^*, \qquad \big( A^* \omega \big)(\mathrm{v}_1, \dots, \mathrm{v}_p) = \omega \big( A\mathrm{v}_1, \dots, A\mathrm{v}_p \big).$$

Notice that for this map, (5.4) still holds.

A particularly interesting case arises when $p = k$ and $W = V$. Indeed, since $\dim \Lambda^k V^* = 1$, $A^* \colon \Lambda^k V^* \to \Lambda^k V^*$ must be the multiplication with a number. To compute this number, let $\mathrm{w} = (\mathrm{w}_1, \dots, \mathrm{w}_k)$ be a $k$-tuple of vectors in $V$. Writing $\mathrm{w} = \mathrm{v} \cdot B$ for some matrix $B$ as above, we obtain

$$A\mathrm{w} = \mathrm{v}\, AB.$$

Here, sligtly abusing notations, I denoted by $A$ on the right hand side of the equality the matrix of the linear map $A$ with respect to the basis $\mathrm{v}$. Hence,

$$\begin{aligned} A^*\big(\mathrm{v}_1^* \wedge \dots \wedge \mathrm{v}_k^*\big)(\mathrm{w}_1, \dots, \mathrm{w}_k) &= \det\big(AB\big) = \det A \cdot \det B \\ &= \det A \cdot \mathrm{v}_1^* \wedge \dots \wedge \mathrm{v}_k^*\,(\mathrm{w}_1, \dots, \mathrm{w}_k) \qquad \Longrightarrow \\ A^*\big(\mathrm{v}_1^* \wedge \dots \wedge \mathrm{v}_k^*\big) &= \det A \cdot \mathrm{v}_1^* \wedge \dots \wedge \mathrm{v}_k^*. \end{aligned}$$

Thus, for $p = \dim V$, $A^*$ is the multiplication with $\det A$. In fact, one could have taken this as the definition of $\det A$ thus avoiding the choice of a basis.

## 5.2 The cotangent bundle

Proceeding just like in Section 4.2, we can construct another manifold starting from $M$. Namely, consider the set

$$T^*M := \bigsqcup_{m\in M} T_m^*M, \qquad \text{where} \qquad T_m^*M := (T_mM)^* = \big\{\chi\colon T_mM \to \mathbb{R} \mid \chi \text{ is linear}\big\}.$$

This comes equipped with a map

$$\pi\colon T^*M \to M, \qquad \pi(\chi) = m \;\Leftrightarrow\; \chi \in T_mM.$$

For example, in the case $M = \mathbb{R}^k$, for each $m \in \mathbb{R}^k$ we have the linear isomorphism

$$A_m\colon \mathbb{R}^k \to T_mM, \qquad A_m y = \sum_{j=1}^k y_j \partial_j\big|_m,$$

cf. (4.1). The dual of this map is also an isomorphism:

$$A_m^*\colon T_m^*M \to \big(\mathbb{R}^k\big)^* \cong \mathbb{R}^k.$$

More explicitly, if $\partial^x|_m$ is the basis of $T_m\mathbb{R}^k$, denote by $dx|_m$ the dual basis, that is

$$dx_i\big|_m\big(\partial_j\big|_m\big) = \delta_{ij}.$$

Then

$$A_m^*\,\chi = y \qquad \Longleftrightarrow \qquad \chi = \sum_{j=1}^k y_j\, dx_j\big|_m = dx\big|_m \cdot y.$$

Hence, just like in the case of the tangent bundle, we obtain

$$T^*\mathbb{R}^k = \bigsqcup_{m\in\mathbb{R}^k} T_m^*\mathbb{R}^k = \mathbb{R}^k \times \mathbb{R}^k \qquad \text{and} \qquad \pi = \pi_1.$$

Furthermore, let $(U, \varphi)$ be a chart on $M$. Write $\varphi = (x_1, \dots, x_k)$ and just like in the case of $\mathbb{R}^k$ define $dx\big|_m$ to be the basis of $T_m^*M$ dual to $\partial^x\big|_m$. Therefore, we obtain the bijection

$$U \times \mathbb{R}^k \to \pi^{-1}(U) = \bigsqcup_{m\in U} T_m^*M, \qquad (m, y) \mapsto \sum_{j=1}^k y_j\, dx_j\big|_m.$$

Combining this with $\varphi\colon U \to \varphi(U)$, which is also a bijection, we obtain finally a bijective map

$$\tau = \tau_\varphi\colon \varphi(U) \times \mathbb{R}^k \to \pi^{-1}(U), \qquad \tau(x,y) := \sum_{j=1}^k y_j\, dx_j|_{\varphi^{-1}(x)}.$$

The following proposition is an analogue of Proposition 4.3 in the present setting. The proof requires cosmetic changes only and is left as an exercise to the reader.

**Proposition 5.5.** *Let $\mathcal{U} := \big\{(U_\alpha, \varphi_\alpha) \mid \alpha \in A\big\}$ be a smooth atlas on $M$. There is a unique second countable and Hausdorff topology on $T^*M$ such that*

$$\mathcal{V} := \Big\{ \big(\pi^{-1}(U_\alpha),\ \tau_\alpha^{-1}\big) \mid \alpha \in A \Big\}$$

*is a $C^0$-atlas on $T^*M$, where $\tau_\alpha := \tau_{\varphi_\alpha}$. This atlas is in fact smooth so that $T^*M$ is a smooth manifold of dimension $2k$. Moreover, $\pi$ is a smooth map with surjective differential at each point.* □

*Remark* 5.6. Part of the proof is to show that the coordinate transformation maps for $\mathcal{V}$ are given by

$$\Theta_{\alpha\beta}(x,y) = \big(\theta_{\alpha\beta}(x),\ \theta_{\alpha\beta\,*}^{\,t}(x)\, y\big).$$

Here I think of $\theta_{\alpha\beta\,*}(x)$ as a Jacobi-matrix of $\theta_{\alpha\beta}$ and the superindex $t$ indicates the transposed matrix. This should be compared with (4.6).

**Definition 5.7.** A smooth map $\omega\colon M \to T^*M$ such that

$$\pi \circ v = id_M \qquad \Longleftrightarrow \qquad \omega(m) \in T_m^*M$$

is called a (smooth) *differential 1-form* on $M$ (or, simply, just 1-form).

Denote

$$\Omega^1(M) := \{\omega \text{ is a smooth differential 1-form on } M\},$$

which has a structure of a $C^\infty(M)$–module with respect to the pointwise addition and multiplication.

Just like in the case of vector fields, any map $\omega\colon M \to T^*M$ such that $\pi \circ \omega = id_M$ admits a coordinate representation. More precisely, this means that for any chart $(U, \varphi)$ on $M$ such that $\varphi = (x_1, \dots, x_k)$ we can write

$$\omega(m) = \sum_{j=1}^k y_j(m)\, dx_j\big|_m = dx\big|_m \cdot y(m), \qquad y\colon U \to \mathbb{R}^k.$$

**Proposition 5.8.** *The map $\omega\colon M \to T^*M$ satisfying $\pi \circ \omega = id_M$ is a smooth vector field if an only if for each chart $(U, \varphi)$ as above the coordinate representation $y$ of $\omega$ is smooth.* □

*Example* 5.9. For any $y \in \mathbb{R}^\ell$ denote $\chi_y \in \big(\mathbb{R}^\ell\big)^*$ the 1-form given by (5.1). If $M \subset \mathbb{R}^\ell$ is a submanifold, then $TM \subset \mathbb{R}^\ell \times \mathbb{R}^\ell$. Define a differential 1-form $\omega$ on $M$ by

$$\omega\big|_m(\mathrm{w}) = \chi_y(\mathrm{w}), \qquad \text{where} \quad \mathrm{w} \in T_mM \subset \mathbb{R}^\ell.$$

For example, choosing $M = S^2 \subset \mathbb{R}^3$ and $y = (1,0,0)$ we obtain

$$\omega(\mathrm{w}) = \mathrm{w}_1, \qquad \text{whenever} \quad \mathrm{w} \in T_mS^2.$$

The reader should check that this yields *smooth* 1-forms.

*Example* 5.10 (The differential of a function). Let $f$ be a smooth function on $M$. For any $m \in M$ the differential of $f$ is a linear map $T_mM \to T_{f(m)}\mathbb{R}$. Recall that we have the isomorphism

$$T_t\mathbb{R} \to \mathbb{R}, \qquad \lambda\frac{d}{dt} \mapsto \lambda,$$

see (4.13). Hence, the composition

$$df\big|_m \colon T_mM \xrightarrow{f_*|_m} T_{f(m)}\mathbb{R} \longrightarrow \mathbb{R}$$

is a linear map, that is $df\big|_m \in T_m^*M$.

Let us compute the coordinate representation of $df$. Thus, if $(U, \varphi)$ is a chart as above, then

$$\begin{aligned} df\big|_m(\partial_j) = \lambda_j \qquad &\iff \qquad f_*\big|_m(\partial_j) = \lambda_j \,\frac{d}{dt}\Big|_{f(m)} \\ &\iff \qquad f_*\big|_m(\partial_j)h = \lambda_j\, h'\big(f(m)\big), \qquad \forall h \in C^\infty(\mathbb{R}) \\ &\implies \qquad f_*\big|_m(\partial_j)h_0 = \lambda_j\, h_0'\big(f(m)\big), \qquad \text{with } \ h_0(t) = t \\ &\implies \qquad \partial_j\big|_m\big(h_0 \circ f\big) = \partial_j\big|_m\big(f\big) = \lambda_j \\ &\implies \qquad \lambda_j = \frac{\partial}{\partial x_j}\Big|_{\varphi(m)} F(x) = \frac{\partial F}{\partial x_j}. \end{aligned}$$

Furthermore, since $(\partial_1, \dots, \partial_k)$ is a basis of $T_mM$, we obtain

$$df = \sum_{j=1}^k \lambda_j dx_j = \sum_{j=1}^k \frac{\partial F}{\partial x_j}\, dx_j.$$

That is, the coordinate representation of $df$ is the map

$$x \mapsto \Big(\frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_k}\Big).$$

In particular, $df$ is a smooth 1-form.

*Example* 5.11. Consider the following special case of the previous example: $M = \mathbb{R}^k$ and $f_j(x) = x_j$. Then

$$df_j = \sum_{i=1}^k \frac{\partial f_j}{\partial x_i}\, dx_i = dx_j.$$

Thus, $dx_j$ is not just an element of the dual basis, but also the differential of the function $x \mapsto x_j$. This explains the choice of notations for the elements of the basis dual to $\partial^x$.

## 5.3 The bundle of $p$-forms

Even slightly more generally than in the previous section, consider the set

$$\Lambda^pT^*M := \bigsqcup_{m \in M} \Lambda^pT_m^*M,$$

which is equipped with the natural projection $\pi \colon \Lambda^pT^*M \to M$. Once again, we can define a Hausdorff second countable topology on this set and a smooth atlas so that $\Lambda^pT^*M$ becomes a smooth manifold. Given a chart $(U,\varphi)$, $\varphi = (x_1,\dots,x_k)$, a map $\omega\colon M \to \Lambda^p T^*M$ such that $\pi\circ\omega = \mathrm{id}_M$ admits a coordinate representation with respect to this chart. In the case $p=2$, this means that there is a unique map $y\colon U \to \mathbb{R}^{\frac{k(k-1)}{2}}$ such that

$$\omega(m) = \sum_{i<j} y_{ij}(m)\, dx_i\big|_m \wedge dx_j\big|_m.$$

Then $\omega$ is smooth if and only if each $y_{ij}$ is smooth. The reader can obtain all these statements just by repeating the arguments used in the case of $TM$ and $T^*M$.

**Definition 5.12.** Let $\omega$ be a smooth $p$-form on $M$. Then the $p$-form $f^*\omega$ defined by

$$f^*\omega\big|_m(\mathrm{v}_1,\dots,\mathrm{v}_p) := \omega\big|_{f(m)}\big(f_*(\mathrm{v}_1),\dots,f_*(\mathrm{v}_p)\big) \qquad \text{for } \mathrm{v}_1,\dots,\mathrm{v}_p \in T_mM$$

is called *the pull-back* of $\omega$ with respect to $f$.

**Proposition 5.13.** *For any two smooth maps $f\colon M \to N$ and $g\colon N \to L$ and any $\omega \in \Omega^p(L)$ we have*

$$\big(g\circ f\big)^*\omega = f^*(g^*\omega).$$

*Proof.* The proof follows directly from Proposition 2.63. Indeed, for any $\mathrm{v}_1,\dots,\mathrm{v}_p \in T_mM$ we have

$$f^*(g^*\omega)(\mathrm{v}_1,\dots,\mathrm{v}_p) = g^*\omega\big(f_*\mathrm{v}_1,\dots,f_*\mathrm{v}_p\big) = \omega\big(g_*f_*\mathrm{v}_1,\dots,g_*f_*\mathrm{v}_p\big) = \big(g\circ f\big)^*\omega(\mathrm{v}_1,\dots,\mathrm{v}_p).$$

□

*Example* 5.14. Consider $dy_j$ as a smooth 1-form on $\mathbb{R}^\ell$. This means that for any $\mathrm{w} \in T_y\mathbb{R}^\ell \cong \mathbb{R}^\ell$ we have $dy_j(\mathrm{w}) = \mathrm{w}_j$. If $f\colon \mathbb{R}^k \to \mathbb{R}^\ell$ is any smooth map, then for $\mathrm{u} \in T_x\mathbb{R}^k \cong \mathbb{R}^k$ we have

$$\big(f^*dy_j\big)(\mathrm{u}) = dy_j\big(f_*\mathrm{u}\big) = \sum_{i=1}^k \frac{\partial f_j}{\partial x_i}\,\mathrm{u}_i = \sum_{i=1}^k \frac{\partial f_j}{\partial x_i}\,dx_i(\mathrm{u}) \qquad \Longrightarrow$$

$$f^*dy_j = \sum_{i=1}^k \frac{\partial f_j}{\partial x_i}\,dx_i = df_j$$

*Example* 5.15. Let $\omega = \sum_{j=1}^\ell \omega_j(y)\,dy_j$ be a 1-form on $\mathbb{R}^\ell$ and $f\colon \mathbb{R}^k \to \mathbb{R}^\ell$ be a smooth map. Just like in the previous example, for any $\mathrm{u} \in T_x\mathbb{R}^k$ we have

$$\big(f^*\omega\big)(\mathrm{u}) = \omega\big|_{f(x)}\big(f_*\mathrm{u}\big) = \sum_{j=1}^\ell \omega_j\big(f(x)\big)dy_j\big(f_*(\mathrm{u})\big) = \sum_{j=1}^\ell \omega_j\big(f(x)\big)df_j(\mathrm{u}) \qquad \Longrightarrow$$

$$f^*\omega = \sum_{j=1}^\ell (\omega_j\circ f)\,df_j = \sum_{j=1}^\ell\sum_{i=1}^k (\omega_j\circ f)\,\frac{\partial f_j}{\partial x_i}\,dx_i.$$

*Example* 5.16. Let $f \in C^\infty(\mathbb{R}^k;\mathbb{R}^k)$. For $\mathrm{w}_1,\dots,\mathrm{w}_k \in T_x\mathbb{R}^k$ we have

$$f^*\big(dx_1\wedge\cdots\wedge dx_k\big)(\mathrm{w}_1,\dots,\mathrm{w}_k) = dx_1\wedge\cdots\wedge dx_k\big(f_*\mathrm{w}_1,\dots,f_*\mathrm{w}_k\big) = \det B, \qquad \text{where}$$

$$B = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_k}\\ \dots & \dots & \dots \\ \frac{\partial f_k}{\partial x_1} & \dots & \frac{\partial f_k}{\partial x_k}\end{pmatrix}\begin{pmatrix} \mathrm{w}_{11} & \dots & \mathrm{w}_{1k}\\ \dots & \dots & \dots\\ \mathrm{w}_{k1} & \dots & \mathrm{w}_{kk}\end{pmatrix} \qquad \Longrightarrow$$

$$f^*\big(dx_1\wedge\cdots\wedge dx_k\big)(\mathrm{w}_1,\dots,\mathrm{w}_k) = \det\big(f_*\big|_x\big)\cdot dx_1\wedge\cdots\wedge dx_k\big(\mathrm{w}_1,\dots,\mathrm{w}_k\big) \qquad \Longrightarrow$$

$$f^*\big(dx_1\wedge\cdots\wedge dx_k\big) = \det\big(f_*\big|_x\big)\cdot dx_1\wedge\cdots\wedge dx_k.$$

L 9

## 5.4 The differential of a 1-form

**Theorem 5.17.** *For any manifold $M$ there is a unique map $d\colon \Omega^1(M) \to \Omega^2(M)$ with the following properties:*

*(i) $d$ is linear, that is*

$$d\big(\lambda_1\omega_1 + \lambda_2\omega_2\big) = \lambda_1 d\omega_1 + \lambda_2 d\omega_2, \qquad \forall\, \lambda_1, \lambda_2 \in \mathbb{R} \ \text{ and } \ \forall\, \omega_1, \omega_2 \in \Omega^1(M);$$

*(ii) $d$ satisfies the Leibniz rule, that is*

$$d\big(f\omega\big) = df \wedge \omega, \qquad \forall\, f \in C^\infty(M) \ \text{ and } \ \forall\, \omega \in \Omega^1(M);$$

*(iii) $d$ commutes with pull-backs, that is*

$$d\big(f^*\omega\big) = f^*\big(d\omega\big), \qquad \forall\, f \in C^\infty(M) \ \text{ and } \ \forall\, \omega \in \Omega^1(M).$$

*(iv) For any smooth function $f$ we have*

$$d(df) = 0.$$

The map $d$ described in the above theorem is called *the exterior differential* (or, simply, *the differential*).

*Sketch of proof.* Assume first that $d$ exists. In the particular case $M = \mathbb{R}^k$, any 1-form $\omega$ can be written as $\sum \omega_j(x)\, dx_j$. Then the linearity and the Leibniz rule yield:

$$\begin{aligned} d\omega = d\Big(\sum_{j=1}^k \omega_j\, dx_j\Big) &= \sum_{j=1}^k d\big(\omega_j\, dx_j\big) = \sum_{j=1}^k \big(d\omega_j \wedge dx_j + \omega_j\, d(dx_j)\big) \\ &= \sum_{j=1}^k d\omega_j \wedge dx_j \end{aligned} \tag{5.18}$$

With this understood, we can use (5.18) *to define* the exterior differential $d\colon \Omega^1(\mathbb{R}^k) \to \Omega^2(\mathbb{R}^k)$. A straightforward, albeit laborious, verification shows, that this map satisfies *(i)–(iv)* indeed.

For a general manifold $M$, we can proceed as follows. Pick a chart $(U_\alpha, \varphi_\alpha)$ and denote $\psi_\alpha := \varphi_\alpha^{-1}\colon \varphi_\alpha(U_\alpha) \to M$. Then for any $\omega \in \Omega^1(M)$ the pull-back $\psi_\alpha^*\omega$ is a 1-form on an open subset of $\mathbb{R}^k$. Denoting temporarily by $d_{\mathbb{R}^k}$ the differential acting on 1-forms on $\mathbb{R}^k$, define

$$d_M\omega\,\big|_{U_\alpha} := \varphi_\alpha^*\Big(d_{\mathbb{R}^k}\big(\psi_\alpha^*\omega\big)\Big).$$

On the overlap $U_{\alpha\beta} = U_\alpha \cap U_\beta$ of two charts, we have

$$\varphi_\alpha^*\Big(d_{\mathbb{R}^k}\big(\psi_\alpha^*\omega\big)\Big) = \varphi_\beta^*\Big(d_{\mathbb{R}^k}\big(\psi_\beta^*\omega\big)\Big) \qquad \Longleftrightarrow \qquad d_{\mathbb{R}^k}\big(\psi_\alpha^*\omega\big) = \psi_\alpha^*\varphi_\beta^*\Big(d_{\mathbb{R}^k}\big(\psi_\beta^*\omega\big)\Big).$$

The letter equality can be established as follows. By Proposition 5.13, we have

$$\psi_\alpha^*\varphi_\beta^*\Big(d_{\mathbb{R}^k}\big(\psi_\beta^*\omega\big)\Big) = \big(\varphi_\beta \circ \psi_\alpha\big)^*\Big(d_{\mathbb{R}^k}\big(\psi_\beta^*\omega\big)\Big) = \theta_{\beta\alpha}^*\Big(d_{\mathbb{R}^k}\big(\psi_\beta^*\omega\big)\Big).$$

Furthermore, using the fact that $d_{\mathbb{R}^k}$ commutes with the pull-backs, we arrive at

$$\theta_{\beta\alpha}^*\Big(d_{\mathbb{R}^k}\big(\psi_\beta^*\omega\big)\Big) = d_{\mathbb{R}^k}\big(\theta_{\beta\alpha}^*\psi_\beta^*\omega\big) = d_{\mathbb{R}^k}\big((\psi_\beta \circ \theta_{\beta\alpha})^*\omega\big) = d_{\mathbb{R}^k}\big(\psi_\alpha^*\omega\big).$$

Hence, $d_M$ is well-defined. It is then straightforward to check that $d_M$ satisfies *(i)–(iv)*. $\square$

## 5.5 Orientability and integration of $k$-forms

**Definition 5.19.** An atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) \mid \alpha \in A\}$ on $M$ such that for all $\alpha, \beta \in A$ the inequality

$$\det \theta_{\alpha\beta\,*} > 0$$

holds everywhere on $\varphi_\beta(U_{\alpha\beta}) \subset \mathbb{R}^k$ is called *oriented*. A maximal oriented atlas is said to be an orientation of $M$. $M$ is called *orientable* if it admits an orientation.

*Example* 5.20.

(a) Since $\mathbb{R}^k$ admits an atlas consisting of just one chart, $\mathbb{R}^k$ is orientable. In fact, the default orientation of $\mathbb{R}^k$ is the maximal oriented atlas containing the chart $(\mathbb{R}^k, id_{\mathbb{R}^k})$.

(b) The atlas on $S^2$ consisting of two charts constructed in the Introduction is *not* oriented. Indeed, by (1.3) we obtain

$$\theta_{SN\,*} = \frac{1}{|y|^4}\begin{pmatrix} y_2^2 - y_1^2 & -2y_1y_2 \\ -2y_1y_2 & y_1^2 - y_2^2 \end{pmatrix} \implies \det \theta_{SN\,*} = -\frac{1}{|y|^8}\big((y_2^2 - y_1^2)^2 + 4y_1^2y_2^2\big) < 0.$$

To obtain an oriented atlas, set

$$\mathcal{V} := \big\{(U_N, \hat\varphi_N),\ (U_S, \varphi_S)\big\},$$

where $\hat\varphi_N = \rho \circ \varphi_N$ and $\rho(y_1, y_2) = (-y_1, y_2)$. Then the coordinate transformation map is given by

$$\hat\theta_{NS} = \rho \circ \theta_{NS} \qquad \implies \qquad \det \hat\theta_{NS\,*} = -\det \theta_{NS} > 0.$$

Hence, $S^2$ is orientable. Moreover, this atlas determines the default orientation of $S^2$. More generally, any $k$-sphere is orientable (oriented).

(c) The product of two orientable (oriented) manifolds is also orientable (oriented). For example, the torus $\mathbb{T}^2$ is oriented.

Let $\mathcal{A}$ be an orientation of $M$. Just like in Example 5.20, (b) fix any linear isomorphism $\rho\colon \mathbb{R}^k \to \mathbb{R}^k$ such that $\det \rho < 0$. Then $\bar{\mathcal{A}} := \big\{(U_\alpha, \rho \circ \varphi_\alpha) \mid \alpha \in A\big\}$ is also an orientation. This is called *the opposite orientation* (to $\mathcal{A}$).

**Exercise 5.21.** Prove that any connected orientable manifold admits exactly two orientations.

A standard example of a non-orientable manifold is the Möbius strip.

Let $M$ be an oriented manifold of dimension $k$. Let $\omega$ be a $k$-form on $M$ such that $\operatorname{supp} \omega$ is a compact subset contained in a chart $(U_\alpha, \varphi_\alpha)$. Then $\psi_\alpha^*\omega$ is a $k$-form on $\mathbb{R}^k$, where $\psi_\alpha = \varphi_\alpha^{-1}$. We can write

$$\psi_\alpha^*\omega = a_\alpha(x_1, \dots, x_k)\, dx_1 \wedge \dots \wedge dx_k,$$

where $\operatorname{supp} a_\alpha$ is a compact subset in $\varphi(U) \subset \mathbb{R}^k$. Define

$$\int_M \omega := \int_{\varphi_\alpha(U_\alpha)} a_\alpha(x_1, \dots, x_k)\, dx_1 \dots dx_k.$$

We wish to show that this is well-defined, that is if $(U_\beta, \varphi_\beta)$ is any other chart from the same oriented atlas such that $\operatorname{supp}\omega \subset U_\beta$, then

$$\int_{\varphi_\alpha(U_\alpha)} a_\alpha(x_1, \dots, x_k)\, dx_1 \dots dx_k = \int_{\varphi_\beta(U_\beta)} a_\beta(y_1, \dots, y_k)\, dy_1 \dots dy_k. \tag{5.22}$$

The computation required to verify this equality is similar to the one we did in the proof of Theorem 5.17. Indeed,

$$\begin{aligned}
&\theta_{\beta\alpha}^* \psi_\beta^* \omega = (\psi_\beta \circ \theta_{\beta\alpha})^* \omega = \psi_\alpha^* \omega && \Longrightarrow \\
&\theta_{\beta\alpha}^* \big( a_\beta\, dy_1 \wedge \dots \wedge dy_k \big) = a_\alpha\, dx_1 \wedge \dots \wedge dx_k && \Longrightarrow \\
&a_\alpha(x) = \big( a_\beta \circ \theta_{\beta\alpha}(x) \big) \cdot \det \theta_{\beta\alpha\,*}(x).
\end{aligned}$$

Applying the change of variables formula in multiple integrals on $\mathbb{R}^k$, we obtain

$$\begin{aligned}
\int_{\varphi_\alpha(U_\alpha)} a_\alpha(x)\, dx &= \int_{\varphi_\beta(U_\beta)} \big( a_\alpha \circ \theta_{\beta\alpha}(y) \big) \cdot \big| \det \theta_{\beta\alpha\,*}(y) \big|\, dy \\
&= \int_{\varphi_\beta(U_\beta)} \big( a_\alpha \circ \theta_{\beta\alpha}(y) \big) \cdot \det \theta_{\beta\alpha\,*}(y)\, dy \\
&= \int_{\varphi_\beta(U_\beta)} a_\beta(y)\, dy,
\end{aligned} \tag{5.23}$$

where $dx_1 \dots dx_k$ is replaced simply by $dx$ to simplify the notations. Notice that the second equality in (5.23) crucially uses the fact that $\det \theta_{\beta\alpha\,*} > 0$. Thus, this establishes (5.22) and, hence, proves that $\int_M \omega$ is well-defined.

More generally, assume that $M$ is a compact oriented manifold. Choose a finite partition of unity $\{\rho_1, \dots, \rho_J\}$ such that $\operatorname{supp}\rho_j$ is contained in a coordinate chart. For any $\omega \in \Omega^k(M)$ this yields the decomposition

$$\omega = \sum_{j=1}^J \omega_j, \qquad \text{where} \quad \omega_j := \rho_j \cdot \omega.$$

Notice that $\operatorname{supp}\omega_j \subset \operatorname{supp}\rho_j$ is contained in a coordinate chart.

**Definition 5.24.** Let $M$ be a compact oriented manifold. For any $\omega \in \Omega^k(M)$ define

$$\int_M \omega := \sum_{j=1}^J \int_M \omega_j.$$

We still need to check that $\int_M \omega$ does not depend on the choice of the partition of unity. To this end, let $\{\nu_i \mid 1 \leq i \leq I\}$ be another finite partition of unity such that $\operatorname{supp}\nu_i$ is contained in some chart for each $i$. We have

$$\sum_{j=1}^J \int_M \omega_j = \sum_{j=1}^J \int_M \Big( \sum_{i=1}^I \nu_i \omega_j \Big) = \sum_{j=1}^J \sum_{i=1}^I \int_M \nu_i \rho_j \omega = \sum_{i=1}^I \int_M \nu_i \omega,$$

where we used the fact that $\{\nu_i\}$ is a partition of unity to obatin the first equality, additivity of the integral in $\mathbb{R}^k$ to obtain the second one, and the fact that $\{\rho_j\}$ is a partition of unity to obtain the last one. The above equality shows that $\int_m \omega$ is well-defined indeed.

*Example* 5.25. For $M = S^1$ choose charts $(U_1, \varphi_1)$ and $(U_2, \varphi_2)$ such that

$$\begin{aligned} &U_1 := S^1 \setminus \{(1,0)\}, && \psi_1 \colon (0, 2\pi) \to U_1, && \psi_1(t) = (\cos t,\ \sin t); \\ &U_2 := S^1 \setminus \{(-1,0)\}, && \psi_2 \colon (-\pi, \pi) \to U_2, && \psi_2(s) = \big(\cos s,\ \sin s\big), \end{aligned}$$

where $\psi_j = \varphi_j^{-1}$. Let $\omega$ be a differential 1-form defined on an open subset $V \subset \mathbb{R}^2$ containing $S^1$. In particular, we can write

$$\omega = a(x,y)\, dx + b(x,y)\, dy$$

for some smooth functions $a$ and $b$ on $V$. Notice that by the compactness of $S^1$ there exists a constant $C > 0$ such that

$$|a(x,y)| \le C \qquad \text{and} \qquad |b(x,y)| \le C \tag{5.26}$$

holds for all $(x, y) \in S^1$.

Furthermore, pick any positive $\varepsilon \ll 1$ and choose a smooth function $\rho_2 \colon S^1 \to [0,1]$ such that $\rho_2 \circ \psi_2 \equiv 1$ on $[-\varepsilon, \varepsilon]$ and $\rho_2 \circ \psi_2 \equiv 0$ on the complement of $[-2\varepsilon, 2\varepsilon]$. Setting also $\rho_1 := 1 - \rho_2$, we obtain a partition of unity subordinate to $\{U_1, U_2\}$.

In order to compute $\int_{S^1} \omega$, it is convenient to compute $\psi_j^* \imath^* \omega = \psi_j^*$ first. Thus,

$$\begin{aligned} \psi_1^*\omega &= \Big(a\big(\cos t, \sin t\big) \cdot (-\sin t) + b\big(\cos t, \sin t\big) \cdot \cos t\Big)\, dt \\ \psi_2^*\omega &= \Big(a\big(\cos s, \sin s\big) \cdot (-\sin s) + b\big(\cos s, \sin s\big) \cdot \cos s\Big)\, ds \end{aligned} \tag{5.27}$$

Therefore,

$$\begin{aligned} \int_{S^1} \imath^*\omega &= \int_{U_1} \rho_1 \omega + \int_{U_2} \rho_2 \omega = \int_0^{2\pi} \big(\rho_1 \circ \psi_1\big) \psi_1^*\omega + \int_{-\pi}^{\pi} \big(\rho_1 \circ \psi_2\big) \psi_2^*\omega \\ &= \int_{2\varepsilon}^{2\pi - 2\varepsilon} \psi_1^*\omega + \int_{[0,2\varepsilon] \cup [2\pi - 2\varepsilon, 2\pi]} \big(\rho_1 \circ \psi_1\big) \psi_1^*\omega + \int_{[-2\varepsilon, 2\varepsilon]} \big(\rho_2 \circ \psi_2\big) \psi_2^*\omega. \end{aligned} \tag{5.28}$$

Using (5.27), (5.26) and $\big|\rho_2 \circ \psi_2\big| \le 1$, for the last term we have

$$\Big| \int_{[-2\varepsilon, 2\varepsilon]} \big(\rho_2 \circ \psi_2\big) \psi_2^*\omega \Big| \le 4C\varepsilon.$$

By a similar argument, the absolute value of the middle term on the right hand side of (5.28) is also bounded by $4C\varepsilon$. Hence, by passing to the limit as $\varepsilon \to 0$, we obtain

$$\int_{S^1} \imath^*\omega = \int_0^{2\pi} \Big(-a\big(\cos t, \sin t\big) \cdot \sin t + b\big(\cos t, \sin t\big) \cdot \cos t\Big)\, dt.$$

*Example* 5.29. Choose the following covering of $S^2$:

$$U_+ := \{(x,y,z) \in S^2 \mid z > 0\}, \quad U_- := \{z < 0\}, \quad \text{and} \quad U_\varepsilon := \{|z| < \varepsilon\}.$$

Arguing just like in the previous example, one can show that for any $\omega \in \Omega^2(S^2)$, we have

$$\int_{S^2} \omega = \int_D \hat{\psi}_N^* \omega + \int_D \psi_S^* \omega,$$

where $\hat{\psi}_N$ and $\psi_S$ are defined in Example 5.20 (b) and $D \subset \mathbb{R}^2$ is the disc of the unit radius.

For example, assume that $\omega = d\eta$ for some $\eta \in \Omega^1(S^2)$. If

$$\psi_S^*\eta = a(x,y)\,dx + b(x,y)\,dy,$$

then applying the Greens formula we obtain

$$\begin{aligned}\int_D \psi_S^* d\eta = \int_D d(\psi_S^*\eta) = \int_D \Big(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}\Big)\,dxdy = \int_{S^1} \big(a\,dx + b\,dy\big)\\ = \int_{[0,2\pi]} \Big(-a\big(\cos t, \sin t\big)\cdot \sin t + b\big(\cos t, \sin t\big)\cdot \cos t\Big)\,dt\end{aligned}$$

Notice that we have

$$f_S := \psi_S\big|_{S^1} = \psi_S \circ \imath_{S^1} \colon S^1 \to S^2, \qquad (x,y) \mapsto (x,y,0),$$

where $\imath_{S^1} \colon S^1 \to D$ is the embedding of the circle as the boundary of the disc. Hence,

$$\begin{aligned} f_S^*\eta = \big(\psi_S \circ \imath_{S^1}\big)^*\eta = \imath_{S^1}^*\big(\psi_S^*\eta\big) \qquad \Longrightarrow\\ \int_{S^1} f_S^*\eta = \int_{[0,2\pi]} \Big(-a\big(\cos t, \sin t\big)\cdot \sin t + b\big(\cos t, \sin t\big)\cdot \cos t\Big)\,dt.\end{aligned}$$

Here the last equality follows by Example 5.25. Thus,

$$\int_D \psi_S^* d\eta = \int_{S^1} f_S^*\eta. \tag{5.30}$$

By a similar argument, we also have

$$\int_D \hat{\psi}_N^* d\eta = -\int_{S^1} f_N^*\eta. \tag{5.31}$$

The minus sign appears for the following reason: when applying Greens formula, one should orient $S^1$ as the boundary of $D$. This yields the opposite orientation of the equator.

Thus, by (5.30) and (5.31) we obtain finally

$$\int_{S^2} d\eta = 0 \qquad \text{for any } \eta \in \Omega^1(S^2).$$

## 5.6 The degree of a map

Let $M$ and $N$ be compact connected oriented manifolds of dimension $k$. Pick any $\omega \in \Omega^k(N)$ such that $\int_N \omega = 1$. The existence of such forms can be established as follows: We can choose a bump function $\rho \in C^\infty(\mathbb{R}^k)$ such that $\operatorname{supp}\rho \subset B_r(0)$ and $a := \int_{B_r(0)} \rho(x)\,dx > 0$. Set $\hat{\omega} := \rho(x)\,dx_1 \wedge \dots \wedge dx_k$ and choose a chart $(U,\varphi)$ on $N$ such that $\varphi(U) \supset B_r(0)$. Having made these choices, we obtain

$$\int_M \varphi^*\hat{\omega} = \int_{B_r(0)} \hat{\omega} = \int_{B_r(0)} \rho(x)\,dx = a > 0,$$

so that for $\omega := a^{-1}\hat{\omega}$ we have $\int_N \omega = 1$.

**Definition 5.32.** Let $f\colon M \to N$ be any smooth map. The number

$$\deg f := \int_M f^*\omega$$

is called *the Brouwer degree* of $f$ (or, simply, the degree of $f$), where $\omega$ is any $k$-form on $N$ such that $\int_N \omega = 1$.

It is by no means obvious that $\deg f$ does not depend on the choice of $\omega$. The proof of this is somewhat technical and is omitted here, however let me outline briefly the main steps required. Assume for the sake of simplicity that $k = 2$ (This is only needed, because the exterior diffrential has been defined for 1-forms, however a suitable generalization of Theorem 5.17 yields the exterior differential as a map $\Omega^p(M) \to \Omega^{p+1}(M)$ for all $p$).

Thus, if $\omega_1$ is another 2-form such that $\int_M \omega_1 = 1$, then for $\omega_0 := \omega - \omega_1$ we have $\int_M \omega_0 = 0$. It is not too hard to show that there exists $\eta \in \Omega^1(M)$ such that $\omega = d\eta$ [BT03, Thm 5.5.5]. Furthermore, one can show that

$$\int_M d\eta = 0 \tag{5.33}$$

holds for any $\omega \in \Omega^1(M)$. This can be seen as follows. Using a partition of unity, write $\eta = \sum_j \eta_j$, where $\operatorname{supp} \eta_j \subset U_j$ and $U_j$ is a chart. Using the Green's formula, one obtains $\int_M d\eta_j = 0$ for each $j$. This in turn implies (5.33).

With this understood, we obtain

$$\omega_1 = \omega - \omega_0 = \omega - d\eta \qquad \Longrightarrow \qquad \int_M \omega_1 = \int_M \omega - \int_M d\eta = \int_M \omega,$$

which shows that $\deg f$ is well-defined indeed.

**Proposition 5.34.** *If $\deg f \neq 0$, then $f$ is surjective.*

*Proof.* Since $M$ is compact and $N$ is Hausdorff, $f(M)$ is closed in $N$, cf. the proof of Corollary 3.19. Hence, if $f$ is non-surjective, then $N \setminus f(M)$ is an open non-empty subset. We can therefore choose $\omega \in \Omega^k(N)$ such that $\int_N \omega = 1$ and $\operatorname{supp} \omega \subset N \setminus f(M)$. Then we must have $\deg f = \int_M f^*\omega = 0$. $\square$

L 10

Let $n \in N$ be a regular value of $f$ (recall that Sard's theorem guarantees the existence of regular values). Since $f^{-1}(n) \subset M$ is closed, this must be compact as a closed subset of a compact space. Since $n$ is a regular value, $f^{-1}(n)$ is in fact a finite set, see the proof of Step 4 of Theorem 2.29, that is

$$f^{-1}(n) = \{\, m_1, \dots, m_\ell \,\}.$$

To any point $m_j$ we assosiate "a sign", that is a number $\varepsilon_j = \pm 1$ as follows. Let $\varphi_\alpha$ be a chart on $M$ centered at $m_j$ and $\psi_\mu$ be a chart on $N$ centered at $n$. If $F_{\alpha\mu}$ is a coordinate representation of $f$ with respect to these charts, then $\det F_{\alpha\mu *}(0) \neq 0$, since $n$ is a regular value. Set

$$\varepsilon_j := \operatorname{sign}\big(\det F_{\alpha\mu *}(0)\big).$$

We wish to show that $\varepsilon_j$ does not depend on the choice of charts. Thus, let $\varphi_\beta$ and $\psi_\nu$ be any other charts from the oriented atlases of $M$ and $N$ respectively. Then the required property follows from the following computation:

$$F_{\beta\nu} = \theta^N_{\nu\mu} \circ F_{\alpha\mu} \circ \theta^M_{\alpha\beta} \qquad \Longrightarrow$$

$$\operatorname{sign}\big(\det F_{\beta\nu *}(0)\big) = \operatorname{sign}\big(\theta^N_{\nu\mu *}(0)\big)\ \operatorname{sign}\big(\theta^M_{\alpha\beta *}(0)\big)\ \operatorname{sign}\big(\det F_{\alpha\mu *}(0)\big) = \operatorname{sign}\big(\det F_{\alpha\mu *}(0)\big)$$

With this understood, we have the following result.

**Theorem 5.35.** *If $n$ is a regular value of $f$, then*

$$\deg f = \sum_{j=1}^{\ell} \varepsilon_j = \sum_{m \in f^{-1}(n)} \varepsilon(m).$$

*In particular,* $\deg f \in \mathbb{Z}$.

*Proof.* Since $n$ is a regular value of $f$, there exists a neighbourhood $V$ of $n$ and neighbourhoods $U_j$ of each $m_j$ such that $f\colon U_j \to V$ is a diffeomorphism, compare with the proof of Step 4 of Theorem 2.29. Without loss of generality, we can assume that each $U_j$ and $V$ are domains of some charts $(U_j, \varphi_j)$ and $(V, \psi)$.

By the argument given at the beginning of this section, there exists a $k$-form on $N$ such that

$$\int_N \omega = 1 \qquad \text{and} \qquad \operatorname{supp} \omega \subset V.$$

Hence, $f^*\omega$ is supported on $U_1 \sqcup \cdots \sqcup U_\ell$. In particular,

$$\deg f = \sum_{j=1}^{\ell} \int_{U_j} f^*\omega$$

Furthermore, denote $\left(\psi^{-1}\right)^*\omega = \eta \in \Omega^k(\mathbb{R}^k)$ and $\xi_j := \psi \circ f\colon U_j \to \psi(V)$. Since $f\colon U_j \to V$ is a diffeomorphism, $(U_j, \xi_j)$ is a chart on $M$. If this chart is positively oriented, then

$$\int_{U_j} f^*\omega = \int_{\xi_j(U_j)} \left(\xi_j^{-1}\right)^* (f^*\omega) = \int_{\xi_j(U_j)} \left(f \circ \xi_j^{-1}\right)^* \omega = \int_{\xi_j(U_j)} \eta = 1.$$

**Exercise 5.36.** Show that if $(U_j, \xi_j)$ is negatively oriented, then

$$\int_{U_j} f^*\omega = -1.$$

By noticing that $(U_j, \xi_j)$ is positively (negatively) oriented if and only if $\varepsilon_j$ is positive (negative), we obtain

$$\int_{U_j} f^*\omega = \varepsilon_j \qquad \Longrightarrow \qquad \deg f = \sum_{j=1}^{\ell} \varepsilon_j,$$

which finishes the proof of this theorem. □

*Remark* 5.37. Theorem 5.35 should be compared with (2.32), where all points were counted with the "+" sign. The reason this worked, is that for any *holomorphic* function each point counts positively indeed, so that (2.32) is the degree of (2.30) in the sense of Definition 5.32. Thus, in some sense Theorem 5.35 is a pretty powerful and far reaching generalization of the proof of Theorem 2.29.

*Remark* 5.38 (A short proof of the fundamental theorem of algebra). With the technology we developed up to this point, the proof of the fundamental theorem of algebra can be made pretty short. Indeed, if $f\colon S^2 \to S^2$ is defined by (2.30), then by Sard's theorem $f$ admits a regular value $n \neq N$. Since $f^{-1}(n) \neq \varnothing$ by the proof of Step 5 of Theorem 2.29, $\deg f$ is positive since each point in $f^{-1}(n)$ counts positively. Hence, $f$ is surjective by Proposition 5.34. This immediately implies that $p$ is surjective. In particular, $p$ has a root.

# Chapter 6

# Further developments

In this chapter I gathered some further developments of the ideas discussed in the previous sections. The proofs are very sketchy (if any) and the reader is advised to check the references for further details.

## 6.1 The hairy ball theorem

An interesting corollary of the ideas, which were discussed in Section 5.6 is the following result.

**Theorem 6.1.** *Any vector field on $S^{2n}$ has at least one zero.*

This theorem is known as *the hairy ball theorem* due to the following formulation: "You can't comb a hairy ball flat without creating a cowlick".

Theorem 6.1 is particularly striking when compared with the following observation: Any odd-dimensional sphere admits a nowhere vanishing vector field. Indeed, an example of such a vector field on $S^{2n+1}$ is given by

$$v\big(x_0, x_1, \dots, x_{2n}, x_{2n+1}\big) = \big(-x_1, x_0, \dots, -x_{2n+1}, x_{2n}\big).$$

**Definition 6.2.** Two smooth maps $f_0, f_1\colon M \to N$ are said to be (smoothly) *homotopic*, if there exists a continuous map $h\colon M \times [0,1] \to N$ such that each $h_t := h\big|_{\{t\}\times M}$ is smooth and

$$h_0 = f_0 \qquad \text{and} \qquad h_1 = f_1.$$

In this case we write $f_0 \simeq f_1$. The map $h$ is called a homotopy between $f_0$ and $f_1$.

The proof of the hairy ball theorem is based on the following simple result, which is of independent interest.

**Lemma 6.3.** *If $f_0$ and $f_1$ are homotopic, then* $\deg f_0 = \deg f_1$.

*Proof.* Let $f_0$ and $f_1$ be two homotopic maps and $h$ be a homotopy. Clearly, the map

$$t \mapsto \int_M h_t^*\omega$$

is a continuous function on $[0,1]$. By Theorem 5.35 this function takes values in the discrete space $\mathbb{Z}$ and therefore must be constant. In particular,

$$\deg f_0 = \int_M h_0^*\omega = \int_M h_1^*\omega = \int_M f_1^*\omega = \deg f_1.$$

$\square$

*Proof of the hairy ball theorem.* The proof consists of the following two steps.

**Step 1.** *The degree of the antipodal map on an even dimensional sphere equals* $-1$.

A direct computation shows that the coordinate representation of the antipodal map with respect to the chart $\big(S^{2n} \setminus \{S\},\ \varphi_S\big)$ is

$$H_1 := \varphi_S \circ h_1 \circ \varphi_S^{-1}, \qquad H_1(y) = -\frac{1}{|y|^2}\, y = -\theta_{SN}(y), \qquad y \in \mathbb{R}^{2n} \setminus \{0\},$$

cf. (1.3). Since $\theta_{SN}$ is a diffeomorphism, for each $z \in \mathbb{R}^{2n} \setminus \{0\}$ there is a unique $y \in \mathbb{R}^{2n} \setminus \{0\}$ such that $H_1(y) = z$. Moreover, by (a generalization of) Example 5.20 (b), we have

$$\det H_{1\,*}(y) = \det \theta_{SN\,*}(y) < 0 \qquad \Longrightarrow \qquad \deg h_1 = -1,$$

where we have used Theorem 5.35.

**Step 2.** *Assume* $S^{2n}$ *admits a nowhere vanishing vector field* $v$. *Then the degree of the atipodal map must equal* $1$.

Think of $v$ as a map $S^{2n} \to \mathbb{R}^{2n+1}$ satisfying

$$\langle v(x), x \rangle = 0 \qquad \text{for all } \ x \in S^{2n}.$$

Since $v$ vanishes nowhere, the map $\hat v(x) := v(x)/\|v(x)\|$ is also a vector field on $S^{2n}$ satisfying $\|\hat v(x)\| = 1$ for all $x \in S^{2n}$. In particular, we can view $\hat v$ as a map $S^{2n} \to S^{2n}$.

Define a map

$$h\colon\, S^{2n} \times [0,1] \to S^{2n} \qquad \text{by} \qquad h(x,t) := x\cos(\pi t) + v(x)\sin(\pi t).$$

Since $\langle \hat v(x), x \rangle = 0$, we have $\|h(x,t)\|^2 = \cos^2(\pi t) + \sin^2(\pi t) = 1$ so that $h$ takes values in $S^{2n}$ indeed.

Furthermore, since

$$h_0(x) = x \qquad \text{and} \qquad h_1(x) = -x,$$

$h$ is a homotopy between $id_{S^{2n}}$ and the antipodal map $h_1$. Hence, by Lemma 6.3 we obtain

$$1 = \deg id_{S^{2n}} = \deg h_1.$$

This finishes the proof of this step and the proof of the hairy ball theorem, since the conclusions of the above two steps contradict each other. □

## 6.2 The Euler characteristic

Let $M$ be an oriented manifold of dimension $k$.

Let $v$ be a vector field on $M$ such that $v(m_0) = 0$ and $v(m) \neq 0$ for all $m \neq m_0$ from some neighbourhood $V$ of $m_0$. In this case we say that $m_0$ is an isolated zero of $v$.

Choose a chart $(U, \varphi)$ centered at $m_0$. The coordinate representation $v = \sum_{j=1}^k v_j(x)\,\partial_j$ yields a map

$$\mathbb{R}^k \to \mathbb{R}^k, \qquad x \mapsto \big(v_1(x), \dots, v_k(x)\big), \tag{6.4}$$

which is well-defined on a closed ball $B_{r_0}(0)$ for some $r_0 > 0$ sufficiently small and vanishes nowhere except at the origin. For any $r \in (0, r_0)$ consider the map

$$h_r\colon S^{k-1} \to S^{k-1}, \qquad h_r(m) = \frac{1}{\sqrt{v_1(rm)^2 + \cdots + v_k(rm)^2}} \big(v_1(rm), \dots, v_k(rm)\big),$$

where $m \in S^{k-1}$. In other words, $h_r$ is essentially the restriction of (6.4) to the sphere of radius $r$ normalized so that this maps into $S^{k-1}$.

**Definition 6.5.** Let $m_0$ be an isolated zero of $v$. The integer

$$I(v, m_0) = \deg h_r$$

is called *the local index* of $v$ at $m_0$.

Notice that by Lemma 6.3, $I(v, m_0)$ does not depend on $r$, since $h_r$ and $h_\rho$ are manifestly homotopic provided $r, \rho \in (0, r_0)$. Also, the local index does not depend on the choice of the chart near $m_0$ [BT03, Lemma 7.3.8].

Assume in addition that $M$ is compact and that $v$ has isolated zeros only. Hence, the number of zeros is finite.

**Theorem 6.6.** *Let $v$ be a vector field on a compact oriented manifold with isolated zeros only. The integer*

$$\chi(M) := \sum_{m \in v^{-1}(0)} I(v, m_0) \tag{6.7}$$

*does not depend on $v$ and is called* the Euler characteristic of $M$. □

The above theorem is a corollary of the so called Poincaré–Hopf theorem [BT03, Thm 7.6.5]. Notice that in the Poincaré–Hopf theorem as stated in [BT03] the Euler characteristic *is defined* as a certain topological invariant of $M$ a priory unrelated to vector fields. I have taken the liberty to define the Euler characteristic by (6.7), which makes obscure that that this number is independent of $v$.

## 6.3 On the classification of manifolds

Let $M_1$ and $M_2$ be two connected oriented manifolds of dimension $k$. Choose $m_j \in M_j$ and a chart $(U_j, \varphi_j)$ centered at $m_j$. Assume that $B_1(0) \subset \varphi_j(U_j)$ and denote $\psi_j := \varphi_j^{-1}\colon B_1(0) \to M_j$. Using the diffeomorphism $B_1(0) \setminus \{0\} \cong S^{n-1} \times (0,1)$, we can vie $\psi_j$ as a diffeomorphism $S^{n-1} \times (0,1) \to \psi_j\big(B_1(0)\big) \setminus \{m_j\}$.

**Definition 6.8.** The space

$$\begin{aligned} &M_1 \# M_2 := \big(M_1 \setminus \{m_1\} \sqcup M_2 \setminus \{m_2\}\big)/\sim, \quad \text{where} \\ &\varphi_1(x, r) \sim \varphi_2(x, 1-r), \qquad x \in S^{n-1} \text{ and } r \in (0,1), \end{aligned}$$

is called *the connected sum* of $M_1$ and $M_2$.

Figure.

It can be shown that $M_1 \# M_2$ is again an oriented manifold of dimension $k$. Moreover, $M_1 \# M_2$ does not depend on the choices involved in the construction (meaning the following:

For any other choice of points $m_j$ and charts $(U_j, \varphi_j)$ as above the results of the above construction are diffeomorphic).

Denote

$$\Sigma_0 = S^2, \quad \Sigma_1 = \mathbb{T}^2, \quad \Sigma_2 = \mathbb{T}^2 \# \mathbb{T}^2, \ \ldots, \quad \Sigma_g = \#_g \mathbb{T}^2. \tag{6.9}$$

The number $g$ is called *the genus* of $\Sigma_g$.

**Theorem 6.10.** *The Euler characteristic of $\Sigma_g$ is given by*

$$\chi(\Sigma_g) = 2 - 2g.$$

□

Notice that the hairy ball theorem follows from Theorems 6.10 and 6.6. In fact, these two theorems imply that any vector field on any surface $\Sigma_g$ has at least one zero except in the case $g = 0$.

**Theorem 6.11** (Classification theorem for compact orientable surfaces)**.** (6.9) *is a complete list of compact connected orientable surfaces (i.e., 2-manifolds). This means that each compact connected orientable surface is diffeomorphic to exactly one surface from* (6.9)*. In particular, $\Sigma_g$ is diffeomorphic to $\Sigma_h$ if and only if $g = h$.* □

Related to this is the following more elementary result.

**Theorem 6.12** (Classification of curves)**.** *Each connected curve (i.e., 1-manifold) is diffeomorphic either to the interval $(0, 1)$ or to the circle $S^1$.*

*Proof.* See [Mil65] or [GP74]. □

A classification of compact manifolds in all dimensions is unknown up to now, however many interesting results are known. Below is a selection of some of those.

**Theorem 6.13** (Milnor'56)**.** *There are smooth 7-manifolds, which are homeomorphic but not diffeomorphic to $S^7$.* □

Later, Kervaire and Milnor showed that there are exactly 28 seven-manifolds, which are homeomorphic but not diffeomorphic to $S^7$. It is even possible to give explicit examples of such manifolds. For example,

$$M_a := \left\{ z \in \mathbb{C}^5 \mid z_1^2 + z_2^2 + z_3^2 + z_4^3 + z_5^{6a-1} = 0 \right\} \cap S_r^9,$$

where $r \ll 1$ and $a = 1, 2, \ldots, 28$, are such examples.

A lot is known about manifolds, which are homeomorphic but not diffeomorphic to $S^n$ for other values of $n$. For example, any three-manifold, which is homeomorphic to $S^3$ must be in fact diffeomorphic to $S^3$. However, it remains unknown up to now, if there is a four-manifold, which is homeomorphic but not diffeomorphic to $S^4$. In contrast, Taubes showed in 1987 that there are uncountably many smooth four-manifolds, which are homeomorphic but not diffeomorphic to $\mathbb{R}^4$. This fascinating story goes, however, far beyond the goals of this course.

## 6.4 The Gauss map

Let $V$ be a finite dimensional vector space. Two bases $\mathrm{v}$ and $\mathrm{w}$ of $V$ are said to be *cooriented* if $\det B > 0$, where $\mathrm{w} = \mathrm{v} \cdot B$. Otherwise, we say that $\mathrm{v}$ and $\mathrm{w}$ have opposite orientations. An orientation of a vector space is a choice of one of the two classes of cooriented bases. For example, the standard basis of $\mathbb{R}^k$ determines a standard orientation of $\mathbb{R}^k$.

If $M$ is an oriented manifold, we can orient each tangent space $T_mM$ as follows. Picking a chart $(U, \varphi)$ from the oriented atlas containing $m$, we obtain a basis

$$\partial(m) = \big(\partial_1, \dots, \partial_k\big) \tag{6.14}$$

of $T_mM$. It follows from the definition of the oriented atlas and (4.4) that any other basis obtained in this way is cooriented with (6.14). Thus, any tangent space of an oriented manifold is oriented.

With this understood, assume $M^2$ is an oriented surface, which is a submanifold of $\mathbb{R}^3$. Then for each $m \in M$ there is a unique vector $\mathrm{n}(m) \in \mathbb{R}^3$ with the following properties: $|\mathrm{n}(m)| = 1$ and $\big(\mathrm{n}(m), \mathrm{v}_1, \mathrm{v}_2\big)$ is an oriented basis of $\mathbb{R}^3$ provided $(\mathrm{v}_1, \mathrm{v}_2)$ is an oriented basis of $T_mM$. In fact, we must have

$$\mathrm{n}(m) = \frac{\mathrm{v}_1 \times \mathrm{v}_2}{|\mathrm{v}_1 \times \mathrm{v}_2|},$$

where $\times$ denotes the cross-product in $\mathbb{R}^3$. It follows that $\mathrm{n}$ depends smoothly on $m$, that is the map

$$G\colon M \to S^2, \qquad G(m) := \mathrm{n}(m)$$

is smooth. This is called *the Gauss map*.

One can show that the following interesting result holds.

**Theorem 6.15** ([BT03, Cor. 7.6.6]). *If $M^2$ is a compact connected oriented surface embedded in $\mathbb{R}^3$, then*

$$\deg G = \chi(M).$$

□

Furthermore, define $\omega \in \Omega^2(S^2)$ as follows: for any pair of vectors $\mathrm{v}_1, \mathrm{v}_2 \in T_mS^2$ set

$$\omega\big|_m(\mathrm{v}_1, \mathrm{v}_2) := dx_1 \wedge dx_2 \wedge dx_3\,\big(G(m), \mathrm{v}_1, \mathrm{v}_2\big),$$

where $\bar G$ is the Gauss map of the standard embedding $S^2 \to \mathbb{R}^3$ (that is, $\bar G(x) = x$). Hence, we obtain

$$\chi(M) = \deg G = \int_M G^*\omega. \tag{6.16}$$

It turns out that the 2-form $G^*\omega \in \Omega^2(M)$ depends only on the "inner geometry" of $M$ in the following sense: Each tangent space $T_mM$ inherits a scalar product $\langle\cdot,\cdot\rangle_m$, which depends smoothly on $m$, that is the Gram matrix of this scalar product with respect to any basis of the form (6.14) has smooth entries. Then $G^*\omega$ depends only on this scalar product but not (directly) on the embedding into $\mathbb{R}^3$. This 2-form is called *the curvature* of $M$ and is a local invariant of $M$, that is its value at $m$ can be computed by knowing the scalar product in any neighbourhood of $m$. On the contrary, $\chi(M)$ characterizes $M$ as a whole object and (6.16) is a beautiful relation between these local and global properties of $M$. This is known as the Gauss-Bonnet theorem, which is one of the most beautiful results in mathematics.

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